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Hochschild (Co)homology of the Dunkl Operator Quantization of $\mathbb{Z}_2$-singularity

Ajay Ramadoss[1] and Xiang Tang[2]

[1]Departement Mathematik, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland and [2]Department of Mathematics, Washington University, St. Louis, MO 63130, USA

Correspondence to be sent to: ajay.ramadoss@math.ethz.ch, xtang@math.wustl.edu

We study Hochschild (co)homology groups of the Dunkl operator quantization of $\mathbb{Z}_2$-singularity constructed by Halbout and Tang. Further, we study traces on this algebra and prove a local algebraic index formula.

1 Introduction

Quantizations of symplectic orbifolds were constructed by many authors [13, 16, 22]. The main idea in the constructions is that the classical quantization methods on a symplectic manifold M can be made equivariantly with respect to the action of a finite group G, and therefore can be generalized to the orbifold M/G. Let us denote the algebra of quantized functions by $\mathbb{A}_{M/G}((\hbar))$. With the efforts of many authors [4, 15, 17–19], we have come to understand the Hochschild and cyclic (co)homology groups of and algebraic index theory on $\mathbb{A}_{M/G}((\hbar))$.

In [11], modeled by the Dunkl operator, Halbout and Tang constructed an algebra $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ as an interesting deformation of the algebra $\mathbb{A}_{M/\mathbb{Z}_2}((\hbar))$ when $G=\mathbb{Z}_2$. This deformation is a global version of a symplectic reflection algebra introduced by Etingof and Ginzburg [7] (also see [6]). We devote this paper to study Hochschild

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(co)homology groups of the algebra $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$. Such study could not only help in understanding the algebra $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$, but also shed light on the relationship between this algebra and the geometry of the singularity. Our hope is that our efforts in this paper could eventually lead to a full generalization of the algebra $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ to general A_n type singularities.

Halbout and Tang's original construction (see [11, Section 4]) of $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ is through a global gluing procedure. Such a global construction makes it difficult to compute the cohomology groups of this algebra. Our solution (Propositions 8 and 9) to this problem is construct a presheaf of algebras on $M/\mathbb{Z}_2$, the space of global sections of which defines the algebra $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$. This crucial improvement gives the key to compute the Hochschild homology of the algebra $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$. It turns out to coincide with shifted Chen–Ruan cohomology groups of $M/\mathbb{Z}_2$ with coefficients in $\mathbb{C}((\hbar_1))((\hbar_2))$. To compute the Hochschild cohomology $\mathrm{HH}^\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2)))$ of the algebra $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$, we were first tempted to apply Van Den Bergh duality [24, 25] in the framework of bornological algebras following [4]. However, it is not clear that even in this context, $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ has a resolution by finitely generated bimodules. We therefore proceed to prove that there exists $\theta \in \mathrm{HH}_{2n}(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2)))$ such that the cap product

$$\theta \cap - : \mathrm{HH}^\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))) \to \mathrm{HH}_{2n-\bullet}(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))),$$

is an isomorphism of $\mathbb{C}((\hbar_1))((\hbar_2))$-modules. This is proved by verifying a similar assertion for the algebra of quantum functions on $M/\mathbb{Z}_2$.

With a good understanding of the Hochschild (co)homology groups of the algebra $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$, we study traces on this algebra, together with a related local algebraic index theorem. Our main new discovery is an explicit trace formula on the Dunkl–Weyl algebra $\mathbb{D}_2((\hbar_1))((\hbar_2))$ generalizing Fedosov's γ-trace [17]. Using this local trace, we define a global trace on the algebra $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ following the idea of [17]. We prove a local index formula for the evaluation of this trace on 1. This local index formula may be viewed as a homological detector of the fact that $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ is a second quantization of the ring of quantum functions on $M/\mathbb{Z}_2$. By the Hochschild homology of $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$, we know that there should be one more trace on this algebra that forms a $\mathbb{C}((\hbar_1))((\hbar_2))$-basis for the space of traces on $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ along with the trace that we have explicitly constructed. Unfortunately, we have not been able to give a local explicit construction of the latter trace. Instead, we succeed in giving an abstract construction of this trace using functors in the derived category of sheaves. It would be very interesting to have a better understanding of this trace, and we plan to come back to this question in the future.

1.1 Organization of this paper

This paper is organized as follows. In Section 2, we study properties of the Dunkl–Weyl algebra $\mathbb{D}_2((\hbar_1))((\hbar_2))$. We compute its Hochschild (co)homology groups. We prove an explicit trace formula on $\mathbb{D}_2((\hbar_1))((\hbar_2))$. In Section 3, we compute the Hochschild (co)homology groups of the algebra $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ with a refined construction of this algebra. By the end of Section 3, we are left with some technical propositions that have to be proved in order to complete the proofs of the main results in this section. In Section 4, we study traces on $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ and prove a local algebraic index formula. We use our knowledge of trace densities to prove the technical propositions required to complete the main results of Section 3.

2 Formal Computations

In this section, we compute the Hochschild homology of formal analogs of the symplectic reflection algebras defined by [7]. As a special case, we obtain the Hochschild homology of the Dunkl–Weyl algebra $\mathbb{D}_2((\hbar_1))((\hbar_2))$ constructed in [11]. We then provide an explicit formula for the unique normalized $\mathbb{C}((\hbar_1))((\hbar_2))$-linear trace ϕ on $\mathbb{D}_2((\hbar_1))((\hbar_2))$. Using ϕ and the Hochschild $2n-2$-cocycle τ_{2n-2} from [10], we construct a Hochschild $2n-2$-cocycle ψ_{2n-2} of the formal Dunkl–Weyl algebra $\mathbb{W}_{n-1}((\hbar_1))\otimes_{\mathbb{C}((\hbar_1))}\mathbb{D}_2((\hbar_1))((\hbar_2))$. We then prove a local Riemann–Roch theorem for a version of this cocycle.

2.1 Hochschild homology

Let G be a finite subgroup of $\mathrm{Sp}(2n,\mathbb{C})$. Then G acts by automorphisms on the Weyl algebra

$$\mathbb{A}_n((\hbar_1)) := \frac{\mathbb{C}\langle x_1,\ldots,x_n,y_1,\ldots,y_n\rangle((\hbar_1))}{\langle [x_i,y_i]=\hbar_1,\ 1\le i\le n;\ [x_i,y_j]=0,\ i\neq j\rangle}.$$

Let a_j denote the number of conjugacy classes of elements of G having eigenvalue 1 with multiplicity j. Let $\mathbb{A}_n^G((\hbar_1))$ denote the subalgebra of elements in $\mathbb{A}_n((\hbar_1))$ fixed by G. One has the following result.

Theorem 1 ([1]).

$$\mathrm{HH}_j(\mathbb{A}_n^G((\hbar_1))) \cong \mathrm{HH}^{2n-j}(\mathbb{A}_n^G((\hbar_1)), \mathbb{A}_n^G((\hbar_1))) \cong \mathbb{C}((\hbar_1))^{a_j}. \qquad \square$$

Standing assumption and notations. We recall that when the triple $(\mathbb{C}^{2n}, \sum_{i=1}^n dx_i \wedge dy_i, G)$ is indecomposable in the sense of [7], then $a_j = 0$ for odd j (see [7, Equation (2.12)]). We shall henceforth assume that this is the case. For notational brevity, we shall henceforth denote $\mathbb{A}_n^G((\hbar_1))$ by $\mathbb{A}_n^G$, keeping in mind that $\mathbb{A}_n^G$ is an algebra over $\mathbb{C}((\hbar_1))$.

In particular, $\mathrm{HH}^2(\mathbb{A}_n^G, \mathbb{A}_n^G)$ is a vector space of dimension a_{2n-2} over $\mathbb{C}((\hbar_1))$. The number a_{2n-2} is the number of conjugacy classes of *symplectic reflections* in G. This is nonzero in general. In other words, $\mathbb{A}_n^G$ has nontrivial formal deformations in general. Let $\theta \in \mathrm{HH}^2(\mathbb{A}_n^G, \mathbb{A}_n^G)$. Then θ defines a $\mathbb{C}((\hbar_1))((\hbar_2))$-linear star product $\star$ on $\mathbb{A}_n^G((\hbar_2))$ such that

$$a \star b = ab + \hbar_2(\dots) \quad \text{for } a, b \in \mathbb{A}_n^G,$$

by [7, Theorem 1.3]. Denote the algebra $(\mathbb{A}_n^G((\hbar_2)), \star)$ by $B^\theta((\hbar_2))$. One has the following generalization of a part of Theorem 1.

Proposition 1.

$$\mathrm{HH}_j(B^\theta((\hbar_2))) \cong \mathbb{C}((\hbar_1))((\hbar_2))^{a_j} \cong \mathrm{HH}^{2n-j}(B^\theta((\hbar_2)), B^\theta((\hbar_2))). \qquad \square$$

Proof. We note that $B^\theta[[\hbar_2]] := (\mathbb{A}_n^G[[\hbar_2]], \star)$ is a subalgebra of $B^\theta((\hbar_2))$. Further, as modules over $\mathbb{C}((\hbar_1))[[\hbar_2]]$,

$$\mathrm{C}_\bullet(B^\theta((\hbar_2))) = \varinjlim_{p \geq 0} \hbar_2^{-p} \mathrm{C}_\bullet(B^\theta[[\hbar_2]]).$$

Therefore, to show that $\mathrm{HH}_j(B^\theta((\hbar_2))) \cong \mathbb{C}((\hbar_1))((\hbar_2))^{a_j}$, it suffices to show that

$$\mathrm{HH}_j(B^\theta[[\hbar_2]]) \cong \mathbb{C}((\hbar_1))[[\hbar_2]]^{a_j}. \tag{1}$$

For this, put $F_p\mathrm{C}_\bullet(B^\theta[[\hbar_2]]) := \hbar_2^{-p}\mathrm{C}_\bullet(B^\theta[[\hbar_2]])$ for $p \leq 0$. This is a complete, increasing exhaustive filtration on $\mathrm{C}_\bullet(B^\theta[[\hbar_2]])$. The completeness of this filtration follows from the facts that $\mathbb{C}[[\hbar_2]] = \varprojlim_{k \geq 0} \frac{\mathbb{C}[[\hbar_2]]}{\hbar_2^k \mathbb{C}[[\hbar_2]]}$ and that $\mathrm{C}_\bullet(B^\theta[[\hbar_2]])$ is a complex of flat $\mathbb{C}[[\hbar_2]]$-modules.

This filtration yields a spectral sequence such that

$$E^1_{pq} = \mathrm{H}_{p+q}\left(\frac{F_p\mathrm{C}_\bullet}{F_{p-1}\mathrm{C}_\bullet}\right) \cong \hbar_2^{-p}\mathbb{C}((\hbar_1))^{a_{p+q}} \quad \text{for } p \leq 0,$$

with $E^1_{pq} = 0$ otherwise. The isomorphism above is a consequence of Theorem 1. Clearly, this spectral sequence is bounded above. Since $a_{p+q} = 0$ for $p+q$-odd, $E^1_{pq} = E^\infty_{pq}$ in this case. By the complete convergence theorem, this spectral sequence converges to $\mathrm{HH}_{p+q}(B^\theta[\![\hbar_2]\!])$. From this, Equation (1) is immediate, proving that $\mathrm{HH}_j(B^\theta((\hbar_2))) \cong \mathbb{C}((\hbar_1))((\hbar_2))^{a_j}$.

The same filtration arguments as above work to prove the statement about Hochschild cohomology. Alternatively, the proof of Theorem 5 applies to this special case to compute the Hochschild cohomology of $B^\theta((\hbar_2))$ from its Hochschild homology. We leave the details of the latter proof to our readers. ■

Remark. Alternatively, by suitably modifying the proof of [4, Proposition 6], we can prove that $B^\theta((\hbar_2))$ is in the class of $VB(2n)$ (See [4, Proposition 2]); then we can obtain the Hochschild cohomology of $B^\theta((\hbar_2))$ from its Hochschild homology using the Van den Bergh duality [24, 25]. □

In what follows, we consider a special example of $B^\theta((\hbar_2))$. Let $n = 1$, $G = \mathbb{Z}_2$ with the generator γ of G acting on $\mathbb{C}\langle x, y\rangle$ by multiplication with -1. Consider the formal symplectic reflection algebra

$$\mathcal{R} := \frac{\mathbb{C}\langle x, y\rangle((\hbar_1))((\hbar_2)) \rtimes \mathbb{Z}_2}{\langle [y, x] = \frac{\hbar_1(1+2\hbar_2\gamma)}{2}\rangle}.$$

Recall that by the PBW theorem there is a vector space isomorphism between $\mathbb{C}[x, y] \rtimes \mathbb{Z}_2((\hbar_1))((\hbar_2))$ and $\mathcal{R}$ identifying the commutative monomial $x^a y^b$ with the image in $\mathcal{R}$ of the (noncommutative) monomial $x^a y^b$. Let $\mathbf{e} := \frac{1+\gamma}{2}$ and let

$$\mathcal{D}_2((\hbar_1))((\hbar_2)) := \mathbb{C}[x, y]^{\mathbb{Z}_2}((\hbar_1))((\hbar_2)).$$

Note that $\mathcal{D}_2((\hbar_1))((\hbar_2))$ is isomorphic (as $\mathbb{C}((\hbar_1))((\hbar_2))$-modules) to the spherical subalgebra $\mathbf{e}\mathcal{R}\mathbf{e}$ of $\mathcal{R}$ as introduced in [7]. Explicitly, this isomorphism identifies $f \in \mathcal{D}_2((\hbar_1))((\hbar_2))$ with $f.\mathbf{e} \in \mathbf{e}\mathcal{R}\mathbf{e}$. As a result, the product on $\mathcal{R}$ induces a star product on $\mathcal{D}_2((\hbar_1))((\hbar_2))$. It was shown in [11] that the star product on $\mathcal{D}_2((\hbar_1))((\hbar_2))$ induces a star product on $\mathrm{C}^\infty(D_\epsilon)^{\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ for any $0 < \epsilon \leq \infty$ (where D_ϵ denotes the open disk of radius ϵ centered at the origin in $\mathbb{R}^2$). This was done by exhibiting an explicit Moyal-type formula for the product on $\mathcal{D}_2((\hbar_1))((\hbar_2))$ [11, Theorem 3.10]. Let $\mathbb{D}_2((\hbar_1))((\hbar_2))$ denote $\mathrm{C}^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ equipped with the star product induced by that on $\mathcal{D}_2((\hbar_1))((\hbar_2))$. The proof of the following proposition is completely analogous to that of a part of Proposition 1. It is therefore omitted. More generally, Proposition 2 holds with $\mathbb{R}^2$ replaced by D_ϵ in the definition of $\mathbb{D}_2((\hbar_1))((\hbar_2))$. We remark that $\mathbb{D}_2((\hbar_1))((\hbar_2))$ is a bornological algebra with a canonical bornology coinciding with both the precompact bornology as well as the von Neumann bornology (see [19, Appendix A] for definitions), and that the tensor powers used to define Hochschild chains of $\mathbb{D}_2((\hbar_1))((\hbar_2))$ are bornological.

Proposition 2.

$$\mathrm{HH}_0(\mathbb{D}_2((\hbar_1))((\hbar_2))) \cong \mathrm{HH}_2(\mathbb{D}_2(\hbar_1))((\hbar_2))) \cong \mathbb{C}((\hbar_1))((\hbar_2)),$$

$$\mathrm{HH}_i(\mathbb{D}_2((\hbar_1))((\hbar_2))) = 0 \quad \text{for } i \neq 0, 2. \qquad \square$$

2.2 The trace formula

Proposition 2 shows that the algebra $\mathbb{D}_2((\hbar_1))((\hbar_2))$ has, up to scalar, a unique trace. For compatibility of our notation with that of [11], we put $z := x + \mathrm{i}y$, $\bar{z} := x - \mathrm{i}y$ where $\mathrm{i} := \sqrt{-1}$. In this subsection, we prove the following explicit formula for a trace on $\mathbb{D}_2((\hbar_1))((\hbar_2))$. For a real number r, let $\lfloor r \rfloor$ denote the greatest integer $\leq r$. For $a \in \mathbb{D}_2((\hbar_1))((\hbar_2))$, let $[a]$ denote the corresponding element in

$$\mathrm{HH}_0(\mathbb{D}_2((\hbar_1))((\hbar_2))) = \frac{\mathbb{D}_2((h_1))((h_2))}{[\mathbb{D}_2((h_1))((h_2)), \mathbb{D}_2((h_1))((h_2))]}.$$

Theorem 2. $[1] \neq 0$ in $\mathrm{HH}_0(\mathbb{D}_2((\hbar_1))((\hbar_2)))$. Let ϕ denote the unique $\mathbb{C}((\hbar_1))((\hbar_2))$-linear trace on $\mathbb{D}_2((\hbar_1))((\hbar_2))$ such that $\phi(1) = 1$. Then

$$\phi(f) = \sum_{k=0}^{\infty} \frac{(\mathrm{i}h_1)^k}{(k!)^2} \cdot \left(\prod_{l=1}^{k} \left\{ \frac{l}{2} + (-1)^{l+1} \frac{2\lfloor \frac{l+1}{2} \rfloor h_2}{l+1} \right\} \right) \cdot \frac{\partial^{2k} f}{\partial z^k \partial \bar{z}^k}(0, 0), \tag{2}$$

for any $f(z, \bar{z}) \in \mathrm{C}^\infty(\mathbb{R}^2)^{\mathbb{Z}_2} \subset \mathbb{D}_2((\hbar_1))((\hbar_2))$. $\square$

Before proving Theorem 2, we remark that Equation (2) can be interpreted as a character formula for the spherical subalgebra $\mathcal{D}_2$ of the formal symplectic reflection algebra $\mathcal{R}$. It would, for instance, be interesting to have analogous formulas for spherical subalgebras of arbitrary formal symplectic reflection algebras. It would then be interesting to compare generalizations of (2) for formal Cherednik algebras with character formulas in [2, 3].

Proof. We first show (2) for any polynomial $f(z,\bar{z})$ in $\mathcal{D}_2((\hbar_1))((\hbar_2))$. We shall perform our computations in $\mathcal{R}$, keeping in mind that $\mathcal{D}_2((\hbar_1))((\hbar_2))$ is the spherical subalgebra $\mathbf{e}.\mathcal{R}.\mathbf{e}$ of $\mathcal{R}$ ($\mathbf{e} := \frac{1+\gamma}{2}$). To begin with,

$$[z,\bar{z}] = \mathrm{i}\hbar_1(1+2\hbar_2\gamma) \quad \text{in } \mathcal{R}. \tag{3}$$

It follows that

$$\begin{aligned}[\bar{z},z^2] &= -\mathrm{i}\hbar_1(1+2\hbar_2\gamma)z - \mathrm{i}\hbar_1.z(1+2\hbar_2\gamma) = -2\mathrm{i}\hbar_1 z - 2\mathrm{i}\hbar_1\hbar_2(\gamma.z+z.\gamma) = -2\mathrm{i}\hbar_1 z\\ &\implies [z^2,\bar{z}] = 2\mathrm{i}\hbar_1 z \quad \text{in } \mathcal{R}.\end{aligned} \tag{4}$$

It also follows that

$$\begin{aligned}[z,\bar{z}^q] &= \sum_{k=1}^{q} \bar{z}^{k-1}\mathrm{i}\hbar_1(1+2\hbar_2\gamma)\bar{z}^{q-k}\\ &= q\mathrm{i}\hbar_1\bar{z}^{q-1} + 2\mathrm{i}\hbar_1\hbar_2\bar{z}^{q-1}\sum_{k=1}^{q}(-1)^{q-k}.\gamma\\ &= q\mathrm{i}\hbar_1\bar{z}^{q-1} + \mathrm{i}\hbar_1\hbar_2\bar{z}^{q-1}(1-(-1)^q)\gamma \quad \text{in } \mathcal{R}.\end{aligned} \tag{5}$$

Therefore,

$$\begin{aligned}[z^2,\bar{z}^p] &= \sum_{k=1}^{p}\bar{z}^{k-1}.2\mathrm{i}\hbar_1 z.\bar{z}^{p-k}\\ \implies \left[\frac{z^2}{2\mathrm{i}\hbar_1},\bar{z}^p\right] &= \sum_{k=1}^{p}\bar{z}^{k-1}.z.\bar{z}^{p-k} = pz\bar{z}^{p-1} - \sum_{k=1}^{p}[z,\bar{z}^{k-1}].\bar{z}^{p-k}\\ &= pz\bar{z}^{p-1} - \sum_{k=1}^{p}(k-1)\mathrm{i}\hbar_1\bar{z}^{p-2} - \sum_{k=1}^{p}\mathrm{i}\hbar_1\hbar_2\bar{z}^{p-2}(1-(-1)^{k-1}).(-1)^{p-k}\gamma\end{aligned}$$

$$= pz\bar{z}^{p-1} - \frac{p(p-1)}{2}\mathrm{i}\hbar_1\bar{z}^{p-2} - \mathrm{i}\hbar_1\hbar_2\bar{z}^{p-2}\gamma.\sum_{k=1}^{p}(1-(-1)^{k-1}).(-1)^{p-k}$$

$$= pz\bar{z}^{p-1} - \frac{p(p-1)}{2}\mathrm{i}\hbar_1\bar{z}^{p-2} - \mathrm{i}\hbar_1\hbar_2 2(-1)^p\left\lfloor\frac{p}{2}\right\rfloor\bar{z}^{p-2}\gamma \quad \text{in } \mathcal{R}. \tag{6}$$

Hence,

$$\left[\frac{z^2}{2\mathrm{i}\hbar_1}, z^{p-2}\bar{z}^p\right] = z^{p-2}\left[\frac{z^2}{2\mathrm{i}\hbar_1}, \bar{z}^p\right]$$
$$= pz^{p-1}\bar{z}^{p-1} - \frac{p(p-1)}{2}\mathrm{i}\hbar_1 z^{p-2}\bar{z}^{p-2} - \mathrm{i}\hbar_1\hbar_2 2(-1)^p\left\lfloor\frac{p}{2}\right\rfloor z^{p-2}\bar{z}^{p-2}\gamma \quad \text{in } \mathcal{R}. \tag{7}$$

Note that z^2 and $z^{p-2}\bar{z}^p$ are elements of $\mathcal{D}_2((\hbar_1))((\hbar_2))$ for any $p \geq 2$. It follows from this and (7) (with $p = k+1$) that

$$[z^k\bar{z}^k] = \left[\mathrm{i}\hbar_1\frac{k}{2}z^{k-1}\bar{z}^{k-1} + \frac{2\mathrm{i}\hbar_1 h_2(-1)^{k+1}\left\lfloor\frac{k+1}{2}\right\rfloor}{k+1}z^{k-1}\bar{z}^{k-1}\gamma\right] \quad \text{in } \mathrm{HH}_0(\mathcal{D}_2((\hbar_1))((\hbar_2))). \tag{8}$$

Note that, for $f, g \in \mathcal{D}_2((\hbar_1))((\hbar_2))$,

$$[f, g]_{\mathcal{D}_2((\hbar_1))((\hbar_2))} = [f.\mathbf{e}, g.\mathbf{e}]_{\mathcal{R}} = [f, g]_{\mathcal{R}}.\mathbf{e}.$$

Since $\gamma.\mathbf{e} = \mathbf{e}$, and each $f \in \mathcal{D}_2((\hbar_1))((\hbar_2))$ is identified with $f.\mathbf{e} \in \mathcal{R}$,

$$[z^k\bar{z}^k] = \mathrm{i}\hbar_1\left(\frac{k}{2} + (-1)^{k+1}\frac{2h_2\left\lfloor\frac{k+1}{2}\right\rfloor}{k+1}\right)[z^{k-1}\bar{z}^{k-1}] \quad \text{in } \mathrm{HH}_0(\mathcal{D}_2((\hbar_1))((\hbar_2))) \text{ for any } k \geq 1. \tag{9}$$

Now recall from [11] that

$$[z\bar{z}, g] = \mathrm{i}\hbar_1\left(z\frac{\partial g}{\partial z} - \bar{z}\frac{\partial g}{\partial \bar{z}}\right). \tag{10}$$

It follows from this that $[z\bar{z}, z^p\bar{z}^q] = \mathrm{i}\hbar_1(p-q)z^p\bar{z}^q$. Hence, for $p \neq q$, $[z^p\bar{z}^q] = 0$ in $\mathrm{HH}_0(\mathcal{D}_2((\hbar_1))((\hbar_2)))$. Hence, for $f \in \mathbb{C}[z,\bar{z}]^{\mathbb{Z}_2} \subset \mathcal{D}_2((\hbar_1))((\hbar_2))$,

$$[f] = \sum_{k=0}^{\infty}\frac{1}{(k!)^2}[z^k\bar{z}^k]\frac{\partial^{2k}f}{\partial z^k\partial\bar{z}^k}(0,0) \quad \text{in } \mathrm{HH}_0(\mathcal{D}_2((\hbar_1))((\hbar_2))). \tag{11}$$

Equation (2) follows from (11) and (9) for $f \in \mathbb{C}[z, \bar{z}]^{\mathbb{Z}_2}$. For arbitrary $f \in C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2} \subset \mathbb{D}_2((\hbar_1))((\hbar_2))$, (2) clearly makes sense. To complete the proof of the desired theorem, we only need to verify that, for arbitrary $f, g \in C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2} \subset \mathbb{D}_2((\hbar_1))((\hbar_2))$, $\phi([f, g]) = 0$, where ϕ is given by (2). Let k be an arbitrary positive integer. By [11, Theorem 3.10] and (2), the coefficients of $\hbar_1^l$ in $\phi([f, g])$ for $l \le k$ depend only on the jets of f and g at $(0, 0)$ of order $\le p(k)$ for some function $p: \mathbb{N} \to \mathbb{N}$. Since ϕ is indeed a trace on $\mathcal{D}_2((\hbar_1))((\hbar_2))$ and since there exists a polynomial $\tilde{f} \in \mathbb{C}[z, \bar{z}]^{\mathbb{Z}_2}$ (resp. $\tilde{g} \in \mathbb{C}[z, \bar{z}]^{\mathbb{Z}_2}$) whose jets up to order $\le p(k)$ at $(0, 0)$ coincide with those of f (resp. g), the coefficients of $\hbar_1^l$ in $\phi([f, g])$ for $l \le k$ vanish. This shows that ϕ extends to a trace on $\mathbb{D}_2((\hbar_1))((\hbar_2))$. ■

Remark. Theorem 2 holds with $\mathbb{R}^2$ replaced by D_ϵ in the definition of $\mathbb{D}_2((\hbar_1))((\hbar_2))$. □

2.3 A Hochschild $2n-2$-cocycle of $\mathbb{W}_{n-1}((\hbar_1)) \otimes_{\mathbb{C}((\hbar_1))} \mathbb{D}_2((\hbar_1))((\hbar_2))$

Let $\mathbb{W}_{n-1}((\hbar_1))$ denote the Weyl algebra

$$\frac{\mathbb{C}\langle p_1, \dots, p_{n-1}, q_1, \dots, q_{n-1}\rangle((\hbar_1))}{\langle [p_i, q_j] = \hbar_1 \delta_{ij} \text{for } 1 \le i, j \le n-1\rangle}.$$

Recall from [10] that $\mathfrak{sp}_{2n-2}$ acts on $\mathbb{W}_{n-1}((\hbar_1))$ by derivations. Introduce coordinates $w_1, \dots, w_{2n-2}$ with $w_{2i-1} = p_i$ and $w_{2i} = q_i$. Suppose $\omega^0 = \sum_i dq_i \wedge dp_i = \frac{1}{2}\sum_{i,j} \omega_{ij}^0 \, dw_i \wedge dw_j$. Then $A = (a_j^i) \in \mathfrak{sp}_{2n-2}$ if and only if the matrix (a_{ij}) is symmetric where $a_{ij} := \sum_{k=1}^{2n-2} \omega_{ik}^0 a_j^k$. The derivation of $\mathbb{W}_{n-1}((\hbar_1))$ corresponding to a matrix A is given by

$$f \mapsto [\tilde{A}, f]_{\hbar_1} \quad \text{where } \tilde{A} = \frac{1}{2}\sum_{i,j} a_{ij} w_i w_j.$$

Similarly, $U(1)$ acts on $\mathbb{D}_2((\hbar_1))((\hbar_2))$ via automorphisms. Explicitly,

$$\exp(\mathrm{i}\theta)(f)(z, \bar{z}) = f(z\exp(\mathrm{i}\theta), \bar{z}\exp(-\mathrm{i}\theta)).$$

Proposition 3. The derivative of the above $U(1)$-action maps $1 \in \mathfrak{u}_1$ to the derivation $[z\bar{z}, -]_{\hbar_1}$ of $\mathbb{D}_2((\hbar_1))((\hbar_2))$. □

Proof. One notes that, for the curve $\theta(t) = t$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\exp(\mathrm{i}t)(f))|_{t=0} = \mathrm{i}z\frac{\partial f}{\partial z} - \mathrm{i}\bar{z}\frac{\partial f}{\partial \bar{z}} = [z\bar{z}, f]_{\hbar_1}.$$

The first equality above is by the chain rule and the second equality above is by Equation (10). ■

Equip $\mathbb{W}_{n-1}((\hbar_1)) \otimes_{\mathbb{C}((\hbar_1))} \mathbb{D}_2((\hbar_1))((\hbar_2))$ with the Lie bracket $[,]_{\hbar_1}$. It follows that one has a map of Lie algebras

$$\begin{aligned} \mathfrak{sp}_{2n-2} \oplus \mathfrak{u}_1 &\hookrightarrow \mathbb{W}_{n-1}((\hbar_1)) \otimes_{\mathbb{C}((\hbar_1))} \mathbb{D}_2((\hbar_1))((\hbar_2)) \\ A \oplus \alpha &\mapsto \frac{1}{2}\sum_{i,j} a_{ij} w_i w_j \otimes 1 + 1 \otimes \alpha z\bar{z}. \end{aligned} \tag{12}$$

Further recall that a $\mathfrak{sp}_{2n-2}$-basic reduced Hochschild cocycle τ_{2n-2} of $\mathbb{W}_{n-1}((\hbar_1))$ is constructed in [10]. Let ψ_{2n-2} be the unique Hochschild $2n-2$-cocycle of $\mathbb{W}_{n-1}((\hbar_1)) \otimes_{\mathbb{C}((\hbar_1))} \mathbb{D}_2((\hbar_1))((\hbar_2))$ such that

$$\psi_{2n-2}(a_0 \otimes b_0, \ldots, a_{2n-2} \otimes b_{2n-2}) = \tau_{2n-2}(a_0, \ldots, a_{2n-2}).\phi(b_0 \star \cdots \star b_{2n-2}), \tag{13}$$

for $a_0, \ldots, a_{2n-2} \in \mathbb{W}_{n-1}((\hbar_1))$ and $b_0, \ldots, b_{2n-2} \in \mathbb{D}_2((\hbar_1))((\hbar_2))$.

Proposition 4. ψ_{2n-2} is $\mathfrak{sp}_{2n-2} \oplus \mathfrak{u}_1$-basic. In other words, for any $\gamma \in \mathfrak{sp}_{2n-2} \oplus \mathfrak{u}_1 \oplus \mathbb{C}((\hbar_1))((\hbar_2))$,

$$\sum_{j=0}^{2n-2} \psi_{2n-2}(a_0 \otimes b_0, \ldots, [\gamma, a_j \otimes b_j]_{\hbar_1}, \ldots, a_{2n-2} \otimes b_{2n-2}) = 0, \tag{14}$$

$$\sum_{j=1}^{2n-2} \psi_{2n-2}(a_0 \otimes b_0, \ldots, \gamma, a_j \otimes b_j, \ldots, a_{2n-3} \otimes b_{2n-3}) = 0, \tag{15}$$

for $a_i \in \mathbb{W}_{n-1}((\hbar_1))$ and $b_i \in \mathbb{D}_2((\hbar_1))((\hbar_2))$. □

Proof. We verify this proposition for the case when $\gamma = \mu \otimes 1$ with $\mu \in \mathfrak{sp}_{2n-2}$ and the case when $\gamma = 1 \otimes \nu$ with $\nu \in \mathfrak{u}_1$. The case when $\gamma \in \mathbb{C}((\hbar_1))((\hbar_2))$ is easy to handle and left to the reader.

For $\gamma = \mu \otimes 1$, $[\gamma, a_j \otimes b_j]_{\hbar_1} = [\mu, a_j]_{\hbar_1} \otimes b_j$. In this case,

$$\sum_{j=0}^{2n-2} \psi_{2n-2}(a_0 \otimes b_0, \ldots, [\gamma, a_j \otimes b_j]_{\hbar_1}, \ldots, a_{2n-2} \otimes b_{2n-2})$$

$$= \sum_{j=0}^{2n-2} \tau_{2n-2}(a_0, \ldots, [\mu, a_j]_{\hbar_1}, \ldots, a_{2n-2}).\phi(b_0 \star \cdots \star b_{2n-2}) = 0, \tag{16}$$

$$\sum_{j=1}^{2n-2} \psi_{2n-2}(a_0 \otimes b_0, \ldots, \gamma, a_j \otimes b_j, \ldots, a_{2n-3} \otimes b_{2n-3})$$

$$= \sum_{j=1}^{2n-2} \tau_{2n-2}(a_0, \ldots, \mu, a_j, \ldots, a_{2n-3}).\phi(b_0 \star \cdots \star b_{2n-3}) = 0. \tag{17}$$

The last equalities in (16) and (17) are because τ_{2n-2} is a $\mathfrak{sp}_{2n-2}$-basic cocycle by [10].

For $\gamma = 1 \otimes \nu$, $[\gamma, a_j \otimes b_j]_{\hbar_1} = a_j \otimes [\nu, b_j]_{\hbar_1}$. In this case,

$$\sum_{j=0}^{2n-2} \psi_{2n-2}(a_0 \otimes b_0, \ldots, [\gamma, a_j \otimes b_j]_{\hbar_1}, \ldots, a_{2n-2} \otimes b_{2n-2})$$

$$= \sum_{j=0}^{2n-2} \tau_{2n-2}(a_0, \ldots, a_{2n-2}).\phi(b_0 \star \cdots \star [\nu, b_j]_{\hbar_1} \star \cdots \star b_{n-2})$$

$$= \tau_{2n-2}(a_0, \ldots, a_{2n-2}).\phi([\nu, b_0 \star \cdots \star b_{2n-2}]_{\hbar_1}) = 0, \tag{18}$$

$$\sum_{j=1}^{2n-2} \psi_{2n-2}(a_0 \otimes b_0, \ldots, \gamma, a_j \otimes b_j, \ldots, a_{2n-3} \otimes b_{2n-3})$$

$$= \sum_{j=1}^{2n-2} \tau_{2n-2}(a_0, \ldots, 1, a_j, \ldots, a_{2n-3}).\phi(b_0 \star \cdots \star \nu \star b_j \star \cdots \star b_{2n-3}) = 0. \tag{19}$$

The last equality in (19) is because the cocycle τ_{2n-2} is normalized (see [10]). ■

2.4 A local Riemann–Roch theorem

For notational brevity, we use $\mathbb{K}$ to denote $\mathbb{C}((\hbar_1))((\hbar_2))$ in this section. Fix an $N \gg n$. Put $R := \mathbb{W}_{n-1} \otimes_{\mathbb{C}((\hbar_1))} \mathbb{D}_2(\hbar_1, \hbar_2))$. Let $\mathfrak{g} := \mathfrak{gl}_N(R)$ and note that $\mathfrak{sp}_{2n-2}(\mathbb{K})$ and $\mathfrak{u}_1(\mathbb{K})$ are Lie subalgebras of $\mathfrak{gl}_n(R)$ via (12) and the diagonal embedding $R \hookrightarrow \mathfrak{gl}_n(R)$. Let $\mathfrak{h} := \mathfrak{sp}_{2n-2}(\mathbb{K}) \oplus \mathfrak{u}_1(\mathbb{K}) \oplus \mathfrak{gl}_N(\mathbb{K})$. By [10, Section 3.1 and 3.2], ψ_{2n-2} may be viewed as a $2n-2$-Lie cocycle $\Psi_{2n-2} \in \mathrm{C}^{2n-2}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{h}; \mathfrak{g}^*)$. Let $\mathrm{ev}_1 : \mathrm{C}^{2n-2}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{h}; \mathfrak{g}^*) \to \mathrm{C}^{2n-2}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$ be the evaluation at the identity. We compute $\mathrm{ev}_1\Psi_{2n-2}$. For this purpose, we closely follow [17, Section 5].

Choose coordinates $p_1, \dots, p_{n-1}, q_1, \dots, q_{n-1}$ of $\mathbb{R}^{2n-2}$ and a coordinate z of $\mathbb{C}^1$. For an element $a \in R$, let a_i denote the homogeneous component of degree i. Note that $a_2 = \sum_{i,j} a_{ij} w_i w_j + \alpha z^2 + \beta z\bar{z} + \lambda \bar{z}^2$ where $w_{2i-1} = p_i$ and $w_{2i} = q_i$. Let $a_2' := a_2 - \alpha.z^2 - \lambda\bar{z}^2$. Then one has a $\mathfrak{h}$-equivariant projection

$$\begin{aligned} \mathrm{pr} &: \mathfrak{g} \to \mathfrak{h}, \\ M \otimes a &\mapsto \frac{\mathrm{tr}(M)}{N} a_2' + M a_0. \end{aligned} \tag{20}$$

Let $C \in \mathrm{Hom}_{\mathbb{K}}(\wedge^2 \mathfrak{g}, \mathfrak{h})$ be the curvature of pr with

$$C(v, w) = [\mathrm{pr}(v), \mathrm{pr}(w)]_{\hbar_1} - \mathrm{pr}([v, w]_{\hbar_1}).$$

One then has a Chern–Weil homomorphism $\chi : (S^\bullet \mathfrak{h}^*)^{\mathfrak{h}} \to \mathrm{C}^{2\bullet}(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$ defined by

$$\chi(P)(v_1 \wedge \cdots \wedge v_{2k}) = \frac{1}{k!} \sum_{\substack{\sigma \in S_{2k} \\ \sigma(2i-1) < \sigma(2i)}} (-1)^\sigma P(C(v_{\sigma(1)}, v_{\sigma(2)}), \dots, C(v_{\sigma(2k-1)}, v_{\sigma(2k)})).$$

Proposition 5. For $N \gg n$, $\chi : (S^q \mathfrak{h}^*)^{\mathfrak{h}} \to \mathrm{H}^{2q}(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$ is an isomorphism for $q \leq n-1$. □

Proof. The proof is almost identical to that of [10, Proposition 5.2]. The only (minor) difference is that, for $p, q \geq 0$,

$$\mathrm{H}^p(\mathfrak{gl}_N(R), S^q \mathfrak{gl}_N(R)^*) = \mathbb{K} \quad \text{when } p = 2n-2 \text{ and } 0 \text{ otherwise.}$$ ■

Form the subspace

$$W_{n-1,1,N} := \left\{ \sum_i f_i p_i \otimes 1 + g z\bar{z} \otimes 1 + \sum_k h_k \otimes M_k \,\middle|\, f_i, g, h_k \in \mathbb{K}[q_1, \dots, q_{n-1}, z^2] \right\}.$$

Proposition 6. $W_{n-1,1,N}$ is a Lie subalgebra of $\mathfrak{gl}_n(R)$. □

Proof. $W_{n-1,1,N}$ is clearly a subspace of $\mathfrak{gl}_n(R)$. The rest follows from direct computations. For example, if $g_1 = \tilde{g}_1(q_1,\ldots,q_{n-1})z^{2k_1}$ and if $g_2 = \tilde{g}_2(q_1,\ldots,q_{n-1})z^{2k_2}$, then

$$[g_1 z\bar{z} \otimes 1, g_2 z\bar{z} \otimes 1]_{\hbar_1} = \tilde{g}_1\tilde{g}_2[z^{2k_1} z\bar{z}, z^{2k_2} z\bar{z}]_{\hbar_1} \otimes 1.$$

Since

$$[z^{2k_1} z\bar{z}, z^{2k_2} z\bar{z}]_{\hbar_1} = z^{2k_1}[z\bar{z}, z^{2k_2}]_{\hbar_1} z\bar{z} - z^{2k_2}[z\bar{z}, z^{2k_2}]_{\hbar_1} z\bar{z} = \mathrm{i}\hbar_1(2k_2 - 2k_1)z^{2k_1+2k_2} z\bar{z},$$

$$[g_1 z\bar{z} \otimes 1, g_2 z\bar{z} \otimes 1]_{\hbar_1} = 2\mathrm{i}(k_2 - k_1)\tilde{g}_1\tilde{g}_2 z^{2k_1+2k_2} z\bar{z} \otimes 1 \in W_{n-1,1,N}.$$

Note that in the above computation we have used Equation (10). ■

Put $\mathfrak{h}_1 = W_{n-1,1,N} \cap \mathfrak{h}$. Then $\mathfrak{h}_1 = \mathfrak{gl}_{n-1}(\mathbb{K}) \oplus \mathfrak{u}_1(\mathbb{K}) \oplus \mathfrak{gl}_N(\mathbb{K})$. Denote by $\tilde{W}_{n-1,1,N}$ the Lie-subalgebra of $W_{n-1,1,N}$ of elements of the form $\sum_i f_i p_i \otimes 1 + g z\bar{z} \otimes 1 + \sum_k h_k \otimes M_k$ where $f_i, g, h_k \in \mathbb{K}[q_1,\ldots,q_{n-1}]$.

Proposition 7. The Chern–Weil homomorphism $\chi : (S^q\mathfrak{h}_1{}^*)^{\mathfrak{h}_1} \to \mathrm{H}^{2q}(W_{n-1,1,N}, \mathfrak{h}_1; \mathbb{K})$ is injective for $q \le n-1$. Further, $\mathrm{H}^{2q}(W_{n-1,1,N}, \mathfrak{h}_1; \mathbb{K}) \cong \mathrm{C}^{2q}(\tilde{W}_{n-1,1,N}, \mathfrak{h}_1; \mathbb{K})$. □

Proof. The proof of [17, Proposition 5.2] goes through in this case as well with minor changes, which we shall point out. $\gamma = \mathrm{Id}$ and $W^{\gamma}_{k,n-k,N}$ is replaced by $W_{n-1,1,N}$. With notation paralleling that in the proof of [17, Proposition 5.2], the basis of $W_{n-1,1,N}$ comprises $q^{\alpha_j} z^{2\beta_j} p_j \otimes 1$, $q^{\alpha} z^{2\beta} z\bar{z} \otimes 1$, and $q^{\alpha_{\mathrm{st}}} z^{2\beta_{\mathrm{st}}} \otimes E_{\mathrm{st}}$ where α, α_j, and α_{st} are multi-indices, $\beta_j, \beta, \beta_{\mathrm{st}} \in \mathbb{N} \cup \{0\}$ and E_{st} denotes an elementary matrix. As in [17], we study the $\mathfrak{h}_1$-action on $\mathrm{Hom}_{\mathbb{K}}(\wedge^\bullet(W_{n-1,1,N}), \mathbb{K})$. The elements $\sigma_1 := \sum_{j=1}^{n-1} q_j p_j \otimes 1$ and $\sigma_2 := z\bar{z} \otimes 1$ act diagonally in this case as well. The formula for the σ_1-action is unchanged from that in [17]. That for the σ_2-action is changed from the corresponding formula in [17] by a factor of $\mathrm{i} := \sqrt{-1}$. We present the formulas here in order to be explicit:

$$[\sigma_1, q^{\alpha_j} z^{2\beta_j} p_j \otimes 1]_{\hbar_1} = \left(\left(\sum_{l=1}^{n-1} \alpha_j^l\right) - 1\right) q^{\alpha_j} z^{2\beta_j} p_j \otimes 1$$

$$[\sigma_1, q^{\alpha} z^{2\beta} z\bar{z} \otimes 1]_{\hbar_1} = \left(\sum_{l=1}^{n-1} \alpha^l\right) q^{\alpha} z^{2\beta} z\bar{z} \otimes 1$$

$$[\sigma_1, q^{\alpha_{\mathrm{st}}} z^{2\beta_{\mathrm{st}}} \otimes E_{\mathrm{st}}]_{\hbar_1} = \left(\sum_{l=1}^{n-1} \alpha_{\mathrm{st}}^l\right) q^{\alpha_{\mathrm{st}}} z^{2\beta_{\mathrm{st}}} \otimes E_{\mathrm{st}}$$

$$[\sigma_2, q^{\alpha_j} z^{2\beta_j} p_j \otimes 1]_{\hbar_1} = 2\mathrm{i}\beta_j q^{\alpha_j} z^{2\beta_j} p_j \otimes 1$$

$$[\sigma_2, q^{\alpha} z^{2\beta} z\bar{z} \otimes 1]_{\hbar_1} = 2\mathrm{i}\beta q^{\alpha} z^{2\beta} z\bar{z} \otimes 1$$

$$[\sigma_2, q^{\alpha_{\mathrm{st}}} z^{2\beta_{\mathrm{st}}} \otimes E_{\mathrm{st}}]_{\hbar_1} = 2\mathrm{i}\beta_{\mathrm{st}} q^{\alpha_{\mathrm{st}}} z^{2\beta_{\mathrm{st}}} \otimes E_{\mathrm{st}}.$$

The rest of the proof proceeds as in the proof of [17, Proposition 5.2] with the obvious modifications. ■

For $X = X_1 \oplus X_2 \oplus X_3 \in \mathfrak{sp}_{2n-2}(\mathbb{K}) \oplus \mathfrak{u}_1(\mathbb{K}) \oplus \mathfrak{gl}_N(\mathbb{K})$, let

$$\hat{A}_{\hbar_1}\mathrm{Ch}_\phi\mathrm{Ch}(X) := \left(\det\left(\frac{\frac{\hbar_1 X_1}{2}}{\sinh\left(\frac{\hbar_1 X_1}{2}\right)}\right)\right)^{\frac{1}{2}} \phi(\exp_\star(X_2))\mathrm{tr}(\exp(X_3)).$$

Here, $\star$ denotes the star product in $\mathbb{D}_2((\hbar_1))((\hbar_2))$ and $X_2 \in \mathfrak{u}_1(\mathbb{K})$ is viewed as an element of $\mathbb{D}_2((\hbar_1))((\hbar_2))$ via Equation (12). We are now in a position to state the local Riemann–Roch.

Theorem 3.

$$[\mathrm{ev}_1\Psi_{2n-2}] = (-1)^{n-1}\chi((\hat{A}_{\hbar_1}\mathrm{Ch}_\phi\mathrm{Ch})_{n-1}). \qquad \square$$

Proof. By Proposition 5, there exists a unique polynomial $Q \in (S^{n-1}\mathfrak{h}^*)^{\mathfrak{h}}$ such that $[\mathrm{ev}_1\Psi_{2n-2}] = \chi(Q)$. Since Q is $\mathfrak{h}$-invariant, it is determined by its value on the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{h}$. The Cartan subalgebra $\mathfrak{a}$ is spanned by the elements $q_i p_i \otimes 1$ for $1 \le i \le n-1$, $z\bar{z} \otimes 1$, and $1 \otimes E_{rr}$ for $1 \le r \le N$. By Propositions 5 and 7, we have the following commutative diagram, whose vertical arrows are injective for $k \le n-1$.

$$\begin{tikzcd}
(S^k\mathfrak{h}^*)^{\mathfrak{h}} \arrow[r] \arrow[d, "\chi"] & (S^k\mathfrak{h}_1^*)^{\mathfrak{h}_1} \arrow[r, "\mathrm{id}"'] \arrow[d, "\chi"] & (S^k\mathfrak{h}_1^*)^{\mathfrak{h}_1} \arrow[d, "\chi"] \\
\mathrm{H}^{2k}(\mathfrak{g}, \mathfrak{h}; \mathbb{K}) \arrow[r] & \mathrm{H}^{2k}(W_{n-1,1,N}, \mathfrak{h}_1; \mathbb{K}) \arrow[r, "\cong"] & \mathrm{C}^{2k}(\tilde{W}_{n-1,1,N}, \mathfrak{h}_1; \mathbb{K}).
\end{tikzcd}$$

Further, $\mathfrak{h}$ and $\mathfrak{h}_1$ have the same Cartan subalgebra $\mathfrak{a}$. Since invariant polynomials are determined by their values on $\mathfrak{a}$, the horizontal arrow at the top left of the above diagram is an injection. It follows that the horizontal arrow at the bottom left of the above diagram is an injection as well. Therefore, in order to determine Q, one may work with the restriction of $\mathfrak{g}$ to $\tilde{W}_{n-1,1,N}$. The identity required to prove the desired theorem then becomes an identity of cocycles rather than an identity at the level of cohomology. Hence, $\mathrm{ev}_1\Psi_{2n-2} = \chi(Q)$ in $\mathrm{C}^{2n-2}(\tilde{W}_{n-1,1,N}, \mathfrak{h}_1; \mathbb{K})$.

We now show that $Q = (-1)^{n-1}(\hat{A}_{\hbar_1}\mathrm{Ch}_\phi\mathrm{Ch})_{n-1}$ as follows. Put

$$u_{ij} := -\tfrac{1}{2}q_i^2 p_i\delta_{ij} + q_iq_jp_j, \quad v_{ir} := q_i \otimes E_{rr}, \quad w_i := q_iz\bar{z}.$$

Note that

$$[p_i, u_{ij}]_{\hbar_1} = p_jq_j, \quad [p_i, v_{ir}]_{\hbar_1} = E_{rr}, \quad [p_i, w_i]_{\hbar_1} = z\bar{z}.$$

Further, $\mathrm{pr}(u_{ij}) = \mathrm{pr}(v_{ir}) = \mathrm{pr}(w_i) = 0$. As in the proof of [10, Lemma 5.3], it follows that

$$(\mathrm{ev}_1\Psi_{2n-2})(p_1 \wedge x_1 \cdots \wedge p_{n-1} \wedge x_{n-1}) = (-1)^{n-1}Q\left(\frac{\partial x_1}{\partial q_1}, \ldots, \frac{\partial x_{n-1}}{\partial q_{n-1}}\right), \tag{21}$$

for any $x_1, \ldots, x_{n-1}$ of the form u_{ij} for $j \le i$, v_{ir}, or w_i. As in the proof of [10, Lemma 5.3], Equation (21) uniquely determines the polynomial Q. Proceeding as in the proof of [10, Lemma 5.3], one shows that Q coincides with the polynomial whose restriction to $\mathfrak{a}$ satisfies

$$\begin{aligned} Q(M_1 \otimes a_1 \otimes b_1, \ldots, M_{n-1} \otimes a_{n-1} \otimes b_{n-1}) &= \mathrm{tr}(M_1 \cdots M_{n-1})\phi(b_1 \star \cdots \star b_{n-1}) \\ \mu_{n-1}\int_{[0,1]^{n-1}} \prod_{1\le i\le j\le n-1} &\mathrm{e}^{\hbar_1\psi(u_i-u_j)\alpha_{ij}} \times (a_1 \otimes \cdots \otimes a_{n-1})\,\mathrm{d}u_1 \ldots \mathrm{d}u_{n-1}. \end{aligned} \tag{22}$$

Here, μ_{n-1} and ψ are exactly as in [10, Section 2.3]. Proceeding exactly as in the proof of [17, Theorem 5.3], one then shows that $Q(Y+Z) = (\hat{A}_{\hbar_1}\mathrm{Ch})(Y).\mathrm{Ch}_\phi(Z)$ for $Y = \sum_i \nu_iq_ip_i + \sum_r \sigma_rE_{rr}$ and $Z = \tau z\bar{z}$ with $\nu_i, \sigma_r, \tau \in \mathbb{K}$. This proves the desired theorem. ■

Remark. We note that every element $X_2 \in \mathfrak{u}_1(\mathbb{K})$ is of the form $\tau z\bar{z}$ for some $\tau \in \mathbb{K}$. Applying Theorem 2, one immediately obtains the following formula for $\mathrm{Ch}_\phi(X_2)$:

$$\mathrm{Ch}_\phi(X_2) = \sum_{k=0}^{\infty}\frac{(\mathrm{i}h_1\tau)^k}{k!}.\prod_{l=1}^{l=k}\left(\frac{l}{2} + (-1)^{l+1}\frac{2\lfloor\frac{l+1}{2}\rfloor h_2}{l+1}\right). \tag{23}$$

□

3 Hochschild Homology of $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$

Let M be a smooth, compact symplectic manifold with a symplectic $\mathbb{Z}_2$-action. Let ω denote the symplectic form on M. Let M_2^γ denote a collection of codimension 2 components of the submanifold M^γ of points fixed by the $\mathbb{Z}_2$-action. In [11], Halbout and Tang construct a quantization $\mathfrak{A}_{M/Z_2}((\hbar_1))((\hbar_2))$ of $\mathbb{A}_{M/\mathbb{Z}_2}((\hbar_1))$, where $\mathbb{A}_{M/\mathbb{Z}_2}((\hbar_1))$ is the $\mathbb{Z}_2$-invariant subalgebra of a quantization of $C^\infty(M)$ with characteristic class ω. In this section, we compute $\mathrm{HH}_\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2)))$ as well as the Hochschild cohomology $\mathrm{HH}^\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2)))$. Starting from this section, we consider $C^\infty(M)$, $\mathbb{A}_{M/\mathbb{Z}_2}((\hbar_1))$, and $\mathfrak{A}_{M/Z_2}((\hbar_1))((\hbar_2))$ as bornological algebras with their canonical bornologies [19, A.6].

3.1 Construction of $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$

To start with, we reformulate the construction $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ of [11] in a way that makes it convenient to compute Hochschild homology. $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ is constructed in three steps. Before proceeding with these steps, we fix a $\mathbb{Z}_2$-invariant almost complex structure on M. This also defines a $\mathbb{Z}_2$-invariant metric on M. Fix any sufficiently small positive real number ϵ. Let B_ϵ denote an ϵ-tubular neighborhood of M_2^γ. Let N denote the normal bundle to M_2^γ. Let N_ϵ denote the ϵ-neighborhood of the zero section of N. We require that ϵ be small enough for us to be able to identify B_ϵ with N_ϵ via the exponential map with respect to some $\mathbb{Z}_2$-invariant metric on N. We fix this exponential map, which we denote by exp.

3.1.1 *Step 1: near M^γ.*

We first explain the construction in the vicinity of M_2^γ. Let P denote the principal $U(1)$-bundle of Hermitian frames on N. Over M_2^γ, one has the *Dunkl–Weyl algebra bundle*

$$\mathcal{V} := P \times_{U(1)} \mathbb{D}_2((\hbar_1))((\hbar_2)).$$

Let $J_1^{\mathrm{sp}} M_2^\gamma$ denote the bundle of symplectic frames on TM_2^γ. One also has a bundle of algebras

$$\mathcal{W} := J_1^{\mathrm{sp}} M_2^\gamma \times_{\mathrm{Sp}(2n-2)} \mathbb{W}_{n-1}((\hbar_1)),$$

over M_2^γ. Halbout and Tang [11] construct a Fedosov connection D on $\wedge^\bullet T^* M_2^\gamma \otimes \mathcal{W} \otimes_{\mathbb{C}((\hbar_1))} \mathcal{V}$. This makes $(\wedge^\bullet T^* M_2^\gamma \otimes \mathcal{W} \otimes_{\mathbb{C}((\hbar_1))} \mathcal{V}, D)$ a sheaf of DG-algebras over M_2^γ. The corresponding sheaf of degree 0 flat sections is a sheaf of algebras over M_2^γ, which we denote by $\mathcal{A}_D$. We note that, for any open subset U of M_2^γ, $\Gamma(U, \mathcal{A}_D) \cong \mathrm{C}^\infty(U, \mathcal{V})$ as vector spaces. Since $\mathbb{D}_2((\hbar_1))((\hbar_2)) \cong \mathrm{C}^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ as vector spaces, $\Gamma(U, \mathcal{A}_D) \cong \mathrm{C}^\infty(U, P \times_{U(1)} \mathrm{C}^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}((\hbar_1))((\hbar_2))) \cong \mathrm{C}^\infty(N|_U)^{\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ as vector spaces. As a result, $\mathrm{C}^\infty(N|_U)^{\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ inherits a new product $\star$ from $\Gamma(U, \mathcal{A}_D)$. Let $\mathcal{C}^\infty(N)$ be the sheaf on M_2^γ such that $\Gamma(U, \mathcal{C}^\infty(N)) = \mathrm{C}^\infty(N|_U)$. The star product $\star$ inherited from $\mathcal{A}_D$ makes $\mathcal{C}^\infty(N)^{\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ a sheaf of algebras on M_2^γ.

Let d denote the ($\mathbb{Z}_2$-invariant) Riemannian metric on M. Consider open subsets on N that are of the form $\exp^{-1}(N_{\alpha,U})$, where

$$N_{\alpha,U} := \{(x \in M | d(x, U) < \alpha\},$$

where U is an open subset of M_2^γ and $0 < \alpha \leq \epsilon$. In what follows, we shall denote $\exp^{-1}(N_{\alpha,U})$ by $N_{\alpha,U}$ itself, since it shall be clear from the context whether we are referring to a tubular neighborhood of U or its image in $N|_U$ via the map $\exp^{-1}$. Since the star product on $\mathbb{D}_2((\hbar_1))((\hbar_2))$ is γ-local, the product on $\mathrm{C}^\infty(N|_U)^{\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ restricts to one on $\mathrm{C}^\infty(N_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2))$. Clearly, the assignment

$$N_{\alpha,U} \mapsto (\mathrm{C}^\infty(N_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \star),$$

is a contravariant functor

$$(\{N_{\alpha,U}\}, \subset) \to \text{algebras}.$$

Near components of M^γ that are not in M_2^γ. On open subsets of M that are of the form $N_{\alpha,U}$, with $\alpha \leq \epsilon$ and U a subset of $M^\gamma \setminus M_2^\gamma$, we make the assignment

$$N_{\alpha,U} \mapsto (\mathrm{C}^\infty(N_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \star_F),$$

where $\star_F$ is the product coming from the Fedosov quantization of $\mathrm{C}^\infty(M \setminus M_2^\gamma)$ with Weyl curvature ω.

3.1.2 *Step 2: away from M^γ*

On $M^- := M \setminus M^\gamma$, things are easy. Note that M^- has a free $\mathbb{Z}_2$-action. The Fedosov quantization $\mathcal{A}_{M^-}((\hbar_1))((\hbar_2))$ of $\mathcal{C}^\infty(M^-)$ with Weyl curvature ω yields a sheaf of algebras on M^- with a star product $\star_F$.

3.1.3 *Step 3: patching*

Let U be an open subset of M_2^γ. Let $N^*_{\alpha,U} := N_{\alpha,U} \setminus \{x | x \in U\}$. Since the product on $\mathbb{D}_2((\hbar_1))((\hbar_2))$ is γ-local, one further has restriction maps

$$(\mathrm{C}^\infty(N_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \star) \to (\mathrm{C}^\infty(N^*_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \star).$$

Further, [11, Proposition 4.1] argues that $(\mathrm{C}^\infty(N^*_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \star)$ is isomorphic as algebras to $(\Gamma(N^*_{\alpha,U}, \mathcal{A}_{M^-}((\hbar_1))((\hbar_2)))^{\mathbb{Z}_2}, \star_F)$. Note that one has a natural inclusion of algebras

$$(\Gamma(N^*_{\alpha,U}, \mathcal{A}_{M^-}((\hbar_1))((\hbar_2)))^{\mathbb{Z}_2}, \star_F) \hookrightarrow (\Gamma(N^*_{\alpha,U}, \mathcal{A}_{M^-}((\hbar_1))((\hbar_2))), \star_F).$$

Further observe that the collection of open subsets $\{U \subset M^- | U \text{open}\} \cup \{N_{\alpha,U} | U \subset M_2^\gamma \text{ is open and } 0 < \alpha \leq \epsilon\}$ forms a base for the topology on M. We denote this base by $\mathcal{B}_\epsilon$. Consider the assignment

$$\begin{aligned} U &\mapsto (\Gamma(U, \mathcal{A}_{M^-}((\hbar_1))((\hbar_2))), \star_F) \text{ for } U \subset M^- \text{open} \\ N_{\alpha,U} &\mapsto (\mathrm{C}^\infty(N_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \star) \text{ for } U \subset M_2^\gamma \\ N_{\alpha,U} &\mapsto (\mathrm{C}^\infty(N_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \star_F) \text{ for } U \subset M^\gamma \setminus M_2^\gamma. \end{aligned} \tag{24}$$

For an open subset V of M^- and an open subset U of M_2^γ, the restriction map $\rho_{N_{\alpha,U},V}$ is defined via the isomorphism from [11, Proposition 4.1] mentioned above. In all other cases, the restriction maps are natural.

Proposition 8. The assignment (24) with restriction maps defined as above gives a $\mathcal{B}_\epsilon$-presheaf of algebras on M. □

Proof. The only nontrivial verification we need to do is to verify the compatibility of restriction maps for a sequence of open subsets of M of the form $W \subset N_{\beta,V} \subset N_{\alpha,U}$, where $V \subset U$ are open subsets of M_2^γ, $\beta \leq \alpha$, and W is an open subset of M^-. The required verification follows from the commutativity of the following diagram where the horizontal arrows are the natural restriction maps:

$$\begin{array}{ccc}
\mathrm{C}^\infty(N_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2)) & \longrightarrow & \mathrm{C}^\infty(N_{\beta,V})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2)) \\
\Big\downarrow & & \Big\downarrow \\
\mathrm{C}^\infty(N^*_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2)) & \longrightarrow & \mathrm{C}^\infty(N^*_{\beta,V})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2)) \\
\Big\downarrow & & \Big\downarrow \\
\Gamma(N^*_{\alpha,U}, \mathcal{A}_{M^-}((\hbar_1))((\hbar_2)))^{\mathbb{Z}_2} & \longrightarrow & \Gamma(N^*_{\beta,V}, \mathcal{A}_{M^-}((\hbar_1))((\hbar_2)))^{\mathbb{Z}_2} \\
\Big\downarrow & & \Big\downarrow \\
\Gamma(N^*_{\alpha,U}, \mathcal{A}_{M^-}((\hbar_1))((\hbar_2))) & \longrightarrow & \Gamma(N^*_{\beta,V}, \mathcal{A}_{M^-}((\hbar_1))((\hbar_2))).
\end{array}$$

The vertical arrows in the topmost square of the above diagram are the natural restriction maps. Those in the middle square are the isomorphisms from [11, Proposition 4.1]. Those in the bottom square are natural inclusions. The bottom square obviously commutes. The commutativity of the top square follows from γ-locality of the product on $\mathbb{D}_2((\hbar_1))((\hbar_2))$. To understand the middle square, we recall the construction of the isomorphism from [11, Proposition 4.1]. This isomorphism is constructed in two steps. To start with, as pointed out in [11, Remark 4.3], one has a $U(1)$-equivariant isomorphism between $\mathbb{D}_2((\hbar_1))((\hbar_2))|_{D^*_\epsilon}$ and $\mathbb{W}_2^{\mathbb{Z}_2}((\hbar_1))((\hbar_2))|_{D^*_\epsilon}$ compatible with restriction to $D^*_{\epsilon'}$ for any $\epsilon' < \epsilon$. This enables the construction of an isomorphism of algebras between $(\mathrm{C}^\infty(N^*_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \star)$ and $(\mathrm{C}^\infty(N^*_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \star_W)$ where the latter algebra is constructed following Section 3.1.1 after replacing $\mathcal{V}$ by $\mathcal{V}_W := P \times_{U(1)} \mathbb{W}_2^{\mathbb{Z}_2}((\hbar_1))((\hbar_2))$. Since the $U(1)$-equivariant isomorphism between $\mathbb{D}_2((\hbar_1))((\hbar_2))|_{D^*_\epsilon}$ and $\mathbb{W}_2^{\mathbb{Z}_2}((\hbar_1))((\hbar_2))|_{D^*_\epsilon}$ compatible with restriction to $D^*_{\epsilon'}$ for any $\epsilon' < \epsilon$, the isomorphism of algebras between $(\mathrm{C}^\infty(N^*_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \star)$ and $(\mathrm{C}^\infty(N^*_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \star_W)$ is compatible with restriction to $N^*_{\beta,V}$ for $V \subset U$ and $\beta \leq \alpha$. One then appeals to [12, Theorem 5.6] to note that since $(\mathrm{C}^\infty(N^*_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \star_W)$ and $\Gamma(N^*_{\alpha,U}, \mathcal{A}_{M^-}((\hbar_1))((\hbar_2)))^{\mathbb{Z}_2}$ are quantizations of $\mathrm{C}^\infty(N^*_{\alpha,U})$ having the same characteristic class, they are isomorphic. The desired proposition therefore follows once we see that the isomorphism from [12, Theorem 5.6] is compatible with restriction to $N^*_{\beta,V}$. This follows from the fact that Kravchenko's construction [12] is a local construction using a Fedosov-type connection. ■

Denote the $\mathcal{B}_\epsilon$-presheaf of algebras on M coming from Proposition 8 by $\mathcal{F}$. One constructs a presheaf of algebras $\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2))$ on M by setting

$$\Gamma(U, \mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2))) := \varprojlim_{\substack{V \subset U \\ V \in \mathcal{B}_\epsilon}} \Gamma(V, \mathcal{F}).$$

Clearly,

$$\Gamma(V, \mathcal{A}_{\text{Dunkl}}((\hbar_1))((\hbar_2))) \cong \Gamma(V, \mathcal{F}) \quad \text{for } V \in \mathcal{B}_\epsilon. \tag{25}$$

Proposition 9.

$$\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2)) \cong \Gamma(M, \mathcal{A}_{\text{Dunkl}}((\hbar_1))((\hbar_2)))^{\mathbb{Z}_2}.$$ □

Proof. An element of $\Gamma(M, \mathcal{A}_{\text{Dunkl}}((\hbar_1))((\hbar_2)))^{\mathbb{Z}_2}$ is equivalent to a pair (α, β) such that $\alpha \in \Gamma(M^-, \mathcal{A}_{\text{Dunkl}}((\hbar_1))((\hbar_2)))^{\mathbb{Z}_2}$, $\beta \in \Gamma(N_{M_2^\gamma,\epsilon}, \mathcal{A}_{\text{Dunkl}}((\hbar_1))((\hbar_2)))$, and $\beta|_{N^*_{M_2^\gamma,\epsilon}}$ coincides with $\alpha|_{N^*_{M_2^\gamma,\epsilon}}$ under the isomorphism of [11, Proposition 4.1]. By (25), this is precisely how elements of $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ are constructed in [11]. That the isomorphism of vector spaces so obtained is an isomorphism of algebras is also immediate from the construction of $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ from [11]. ■

3.1.4 *Some important recollections and observations*

We recall that the algebra $C^\infty(M)$ comes equipped with a canonical bornology (see [19, Appendix A6]). As a result, the algebra $C^\infty(M)^{\mathbb{Z}_2}$ is also equipped with the subspace bornology (which we shall still call canonical). This induces bornologies on $\mathbb{A}_{M/\mathbb{Z}_2}((\hbar_1))$ and $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$. Similarly, sections of the presheaf $\mathcal{A}_{\text{Dunkl}}((\hbar_1))((\hbar_2))$ (and other similarly constructed presheaves described in the following paragraphs) over each open $V \subset M$ are equipped with the canonical bornology. The isomorphism in Proposition 9 is an isomorphism of bornological $\mathbb{C}((\hbar_1))((\hbar_2))$-modules.

Further, note that for $U \subset M_2^\gamma$, U open, the product $\star$ constructed in Section 3.1.1 on $C^\infty(N_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ restricts to a product on $C^\infty(N_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))[\![\hbar_2]\!]$. This is because the star product on $C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ coming from $\mathbb{D}_2((\hbar_1))((\hbar_2))$ restricts to a product on $C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}((\hbar_1))[\![\hbar_2]\!]$. One also notes that the restriction map from $C^\infty(N_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ to $\Gamma(N^*_{\alpha,U}, \mathcal{A}_{M^-}((\hbar_1))((\hbar_2)))$ maps $C^\infty(N_{\alpha,U})^{\mathbb{Z}_2}((\hbar_1))[\![\hbar_2]\!]$ to $\Gamma(N^*_{\alpha,U}, \mathcal{A}_{M^-}((\hbar_1))[\![\hbar_2]\!])$. This is immediate once we note that the isomorphism between $\mathbb{D}_2((\hbar_1))((\hbar_2))|_{D^*_\epsilon}$ and $\mathbb{W}_2^{\mathbb{Z}_2}((\hbar_1))((\hbar_2))|_{D^*_\epsilon}$ used to construct the above restriction map in [11, Proposition 4.1] is of the form $\text{id} + \hbar_2\nu_1 + \text{higher order terms in } \hbar_2$.

One may therefore follow Section 3.1 (Sections 3.1.1–3.1.3) to construct a sub-presheaf $\mathcal{A}_{\text{Dunkl}}((\hbar_1))[\![\hbar_2]\!]$ of the presheaf of algebras $\mathcal{A}_{\text{Dunkl}}((\hbar_1))((\hbar_2))$ constructed in Sections 3.1.1–3.1.3. We denote $\Gamma(M, \mathcal{A}_{\text{Dunkl}}((\hbar_1))[\![\hbar_2]\!])^{\mathbb{Z}_2}$ by $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))[\![\hbar_2]\!]$. We may view $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))[\![\hbar_2]\!]$ as a filtered algebra with $F_{-p}\mathfrak{A}_{M/\mathbb{Z}_2} := \hbar_2^p\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))[\![\hbar_2]\!]$ for $p \geq 0$. The associated graded algebra with respect to this filtration is clearly $\mathbb{A}_{M/\mathbb{Z}_2}((\hbar_1))[\![\hbar_2]\!]$. Similarly, the presheaf $\mathcal{A}_{\text{Dunkl}}((\hbar_1))[\![\hbar_2]\!]$ may be viewed as a filtered presheaf of algebras with $F_{-p}\mathcal{A}_{\text{Dunkl}}((\hbar_1))[[\hbar_2]] := \hbar_2^p\mathcal{A}_{\text{Dunkl}}((\hbar_1))[[\hbar_2]]$ for $p \geq 0$. The associated graded presheaf of algebras with respect to this filtration is simply the presheaf $\mathcal{A}((\hbar_1))[[\hbar_2]]$ constructed following Sections 3.1.1–3.1.3 with $\mathbb{D}_2((\hbar_1))((\hbar_2))$ replaced by $\mathbb{W}_2^{\mathbb{Z}_2}((\hbar_1))[[\hbar_2]]$. One notes that $\Gamma(M, \mathcal{A}((\hbar_1))[[\hbar_2]])^{\mathbb{Z}_2} \cong \mathbb{A}_{M/\mathbb{Z}_2}((\hbar_1))[[\hbar_2]]$.

3.2 Computation of Hochschild homology

For the rest of the paper, any Hochschild chain complex of an algebra is a Hochschild chain complex of a bornological algebra. The algebras we are working with are all equipped with canonical bornologies (which coincide with their precompact as well as with their von Neumann bornologies: see [19, Appendix A] for definitions). Recall that if A is a bornological algebra, the (bornological) space of Hochschild k-chains of A is the $k+1$-st bornological tensor power of A. The reader may refer to [19, Appendix A] for an excellent introduction to the homological algebra of bornological algebras and modules. Clearly, the assignment

$$U \mapsto \mathrm{C}_\bullet(\Gamma(U, \mathcal{A}_{\text{Dunkl}}((\hbar_1))((\hbar_2)))),$$

gives a complex of presheaves of $\mathbb{C}((\hbar_1))((\hbar_2))$-modules on M. We denote this complex of presheaves by $\mathrm{C}_\bullet(\mathcal{A}_{\text{Dunkl}}((\hbar_1))((\hbar_2)))$.

Proposition 10. The natural map from global sections to hypercohomology

$$\mathrm{HH}_\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))) \to \mathbb{H}^{-\bullet}(M, \mathrm{C}_\bullet(\mathcal{A}_{\text{Dunkl}}((\hbar_1))((\hbar_2)))^{\mathbb{Z}_2},$$

is an isomorphism of $\mathbb{C}((\hbar_1))((\hbar_2))$-modules. □

Proof. It is clear that

$$\mathrm{C}_\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))) = \varinjlim_{p\geq 0} \hbar_2^{-p}\mathrm{C}_\bullet((\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))[[\hbar_2]], \star)).$$

Further, as complexes of presheaves on M,

$$\mathrm{C}_\bullet(\mathcal{A}_{\text{Dunkl}}((\hbar_1))((\hbar_2))) \cong \varinjlim_{p\geq 0} \hbar_2^{-p}\mathrm{C}_\bullet((\mathcal{A}_{\text{Dunkl}}((\hbar_1))[[\hbar_2]], \star)).$$

Further, the above isomorphism of complexes of presheaves is $\mathbb{Z}_2$-equivariant. Since cohomology commutes with direct limits, the desired proposition follows from Proposition 11, which follows. ■

Proposition 11. The natural map

$$\mathrm{HH}_\bullet((\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))[\![\hbar_2]\!],\star)) \to \mathbb{H}^{-\bullet}(M, \mathrm{C}_\bullet((\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))[\![\hbar_2]\!],\star)))^{\mathbb{Z}_2},$$

is an isomorphism of $\mathbb{C}((\hbar_1))[\![\hbar_2]\!]$-modules. □

Proof. Pick a $\mathbb{Z}_2$-invariant finite cover $\mathfrak{V}$ of M by sufficiently small open sets. Then $\mathbb{H}^{-\bullet}(M, \mathrm{C}_\bullet((\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))[\![\hbar_2]\!],\star)))^{\mathbb{Z}_2}$ is the cohomology of the complex of $\mathbb{Z}_2$-invariant cochains in

$$\mathrm{Tot}^{\oplus}(\hat{C}_{\mathfrak{V}}(\mathrm{C}_\bullet((\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))[\![\hbar_2]\!],\star)))).$$

Note that $\mathrm{C}_\bullet((\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))[\![\hbar_2]\!],\star))$ is a filtered complex with

$$F_{-p}\mathrm{C}_\bullet((\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))[\![\hbar_2]\!],\star)) := \hbar_2^p\mathrm{C}_\bullet((\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))[\![\hbar_2]\!],\star)),$$

for $p \geq 0$. Similarly, $\mathrm{C}_\bullet((\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))[\![\hbar_2]\!],\star))$ is a filtered complex with

$$F_{-p}\mathrm{C}_\bullet((\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))[\![\hbar_2]\!],\star)) := \hbar_2^p\mathrm{C}_\bullet((\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))[\![\hbar_2]\!],\star)),$$

for $p \geq 0$. The natural map

$$\mathrm{C}_\bullet((\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))[\![\hbar_2]\!],\star)) \to \mathrm{Tot}^{\oplus}(\hat{C}_{\mathfrak{V}}(\mathrm{C}_\bullet((\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))[\![\hbar_2]\!],\star))))^{\mathbb{Z}_2},$$

is compatible with filtrations, the filtration on the right-hand side being induced by the filtration on $\mathrm{C}_\bullet((\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))[\![\hbar_2]\!],\star))$. The E^1_{pq}-terms of the corresponding spectral sequences are $\hbar_2^{-p}\mathrm{HH}_{p+q}(\mathbb{A}_{M/\mathbb{Z}_2}((\hbar_1)))$ and $\hbar_2^{-p}\mathrm{H}_{p+q}(\mathrm{Tot}^{\oplus}(\hat{C}_{\mathfrak{V}}(\mathrm{C}_\bullet(\mathcal{A}((\hbar_1)))))^{\mathbb{Z}_2})$, respectively. The spectral sequences on both sides converge by the complete convergence theorem. It therefore remains to verify that the induced map on E^1_{pq} terms described above is an isomorphism. Let $\mathcal{C}_p$ denote the $\mathcal{B}_\epsilon$-presheaf of algebras such that $\mathcal{C}_p(V) = \mathrm{C}^\infty(V)$ and $\mathcal{C}_p(N_{\alpha,U}) = \mathrm{C}^\infty(N_{\alpha,U})^{\mathbb{Z}_2}$. As in the step immediately after Proposition 8, we construct a presheaf of algebras out of $\mathcal{C}_p$, which we shall continue to denote by $\mathcal{C}$. An argument essentially identical to that used to reduce Proposition 10 to the above verification can be further used to reduce the above verification to a verification that the natural map between $\mathrm{HH}_\bullet(\mathrm{C}^\infty(M/\mathbb{Z}_2))$ and $\mathrm{H}_\bullet(\mathrm{Tot}^\oplus(\hat{\mathcal{C}}_{\mathfrak{V}}(\mathrm{C}_\bullet(\mathcal{C})))^{\mathbb{Z}_2})$ is an isomorphism. Let $\pi : M \to M/\mathbb{Z}_2$ denote the natural projection. Note that the cover $\mathfrak{V} := \{V_\alpha\}$ of M is such that the open subsets $\pi(V_\alpha)$ form an open cover $\mathfrak{W}$ of $M/\mathbb{Z}_2$ (of course, if $\pi(V_\alpha) = \pi(V_\beta)$ for some $\alpha \neq \beta$, we count $\pi(V_\alpha)$ only once in $\mathfrak{W}$). Note that the homology $\mathrm{H}_\bullet(\mathrm{Tot}^\oplus(\hat{\mathcal{C}}_{\mathfrak{V}}(\mathrm{C}_\bullet(\mathcal{C})))^{\mathbb{Z}_2})$ coincides with $\mathrm{H}_\bullet(\mathrm{Tot}^\oplus(\hat{\mathcal{C}}_{\mathfrak{W}}(\mathrm{C}_\bullet(\mathcal{C}^\infty_{M/\mathbb{Z}_2}))))$. The latter is precisely the hypercohomology $\mathbb{H}^{-\bullet}(M/\mathbb{Z}_2, \mathrm{C}_\bullet(\mathcal{C}^\infty_{M/\mathbb{Z}_2}))$. Since $\mathrm{C}_\bullet(\mathcal{C}^\infty_{M/\mathbb{Z}_2})$ is a complex of fine presheaves on $M/\mathbb{Z}_2$, the hypercohomology $\mathbb{H}^{-\bullet}(M/\mathbb{Z}_2, \mathrm{C}_\bullet(\mathcal{C}^\infty_{M/\mathbb{Z}_2}))$ is precisely the homology $\mathrm{H}_\bullet(\Gamma(M/\mathbb{Z}_2, \mathrm{C}_\bullet(\mathcal{C}^\infty_{M/\mathbb{Z}_2})))$ of the complex of global sections of $\mathrm{C}_\bullet(\mathcal{C}^\infty_{M/\mathbb{Z}_2})$. The Hochschild k-chains in this complex are germs along the principal diagonal $X \to X^k$ of smooth functions on X^k, where $X = M/\mathbb{Z}_2$. We are, therefore, reduced to showing that the Hochschild complexes $\mathrm{C}_\bullet(\mathrm{C}^\infty_{M/\mathbb{Z}_2})$ and $\Gamma(M/\mathbb{Z}_2, \mathrm{C}_\bullet(\mathcal{C}^\infty_{M/\mathbb{Z}_2}))$ give the same homology. The localization argument in [23, Section 3], which works for $M/\mathbb{Z}_2$ as well, does exactly this. ■

We postpone the proof of the following proposition to Section 4. Let $i_l : M^\gamma_{2l} \to M$ denote the natural inclusion from the collection M^γ_{2l} of codimension $2l$ components of M^γ into M.

Proposition 12. As complexes of presheaves on M, $\mathrm{C}_\bullet(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2))$ is quasi-isomorphic to $\underline{\mathbb{C}}((\hbar_1))((\hbar_2))[2n] \bigoplus \oplus_{l\geq 1} i_{l*}\underline{\mathbb{C}}((\hbar_1))((\hbar_2))[2n-2l]$, where $\underline{\mathbb{C}}((\hbar_1))((\hbar_2))$ is the sheaf of locally constant $\mathbb{C}((\hbar_1))((\hbar_2))$-valued functions. □

The following theorem is an immediate consequence of Propositions 10 and 12.

Theorem 4.

$$\mathrm{HH}_\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))) \cong (\mathrm{H}^{2n-\bullet}(M, \mathbb{C})^{\mathbb{Z}_2} \bigoplus \oplus_{l\geq 1} \mathrm{H}^{2n-2l-\bullet}(M^\gamma_{2l}, \mathbb{C}))((\hbar_1))((\hbar_2)). \tag{26}$$

□

We remark that the right-hand side of (26) is the Chen–Ruan (or "stringy") cohomology $\mathrm{H}^{2n-\bullet}_{\mathrm{CR}}(M/\mathbb{Z}_2, \mathbb{C})((\hbar_1))((\hbar_2))$ of the orbifold $M/\mathbb{Z}_2$.

3.3 Hochschild cohomology of $\mathfrak{A}_{M/\mathbb{Z}_2}$

Given Theorem 4, one is initially tempted to follow the Van Den Bergh duality arguments in [4, Proposition 6] and then appeal to Van Den Bergh duality to compute

$\mathrm{HH}^\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2)))$. However, even in the bornological setting, it is not clear that $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ has a bimodule resolution by finitely generated projective $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$-bimodules. We therefore adopt a different method inspired by [14] to show that one has a Van Den Bergh duality-type isomorphism relating Hochschild cohomology to homology in our situation.

Let $A := \mathbb{A}_M((\hbar))$ be the Fedosov quantization of $\mathrm{C}^\infty(M)$ with Weyl curvature ω. Consider the algebra $B := A \rtimes \mathbb{Z}_2$. Following [5, 17], one can construct trace density maps

$$\chi_l : \mathrm{C}_\bullet(B) \to \Omega^{2n-2l-\bullet}_{M^\gamma_{2l}},$$

for each $l \geq 0$. Here, for $l > 1$, M^γ_{2l} denotes the collection of components of M^γ of codimension $2l$. By convention, $M^\gamma_0 := M$. Recall, for instance, from [4] that $\mathrm{HH}_\bullet(B) = (\mathrm{HH}_\bullet(A) \oplus \mathrm{HH}_\bullet(A, A_\gamma))^{\mathbb{Z}_2}$. The map induced on homologies by χ_0, after this identification, coincides with the map induced on homologies by the trace density χ_{FFS} that one may construct in the situation of [10] following [5]. We also remark that the constructions of the χ_l are local in nature on $X := M/\mathbb{Z}_2$. Let θ be a Hochschild $2n$-cycle of B such that $\chi_0(\theta) = 1$. The cap product with θ gives a map of complexes

$$\theta \cap - : \mathrm{C}^\bullet(B, B) \to \mathrm{C}_{2n-\bullet}(B).$$

Proposition 13. $\theta \cap -$ is a quasi-isomorphism. □

We remark that the choice of the Hochschild $2n$-cycle θ is not unique. Any $2n$-cycle of B with $\chi_0(\theta) = 1$ suffices.

Proof. We divide the proof into two steps.
Step 1: Recall that $X := M/\mathbb{Z}_2$. Let $\mathcal{C}_{B,\bullet}$ denote the chain complex of sheaves on X whose k-chains are given by the sheaf associated with the presheaf

$$U \mapsto \mathrm{C}_k(\mathbb{A}(\pi^{-1}(U)) \rtimes \mathbb{Z}_2).$$

Here, $\pi : M \to X$ is the natural projection. Let $\mathcal{C}^\bullet_B$ denote the cochain complex of sheaves (with Hochschild coboundary) whose sheaf of k-cochains is the sheafification of the presheaf

$$U \mapsto \mathrm{C}^k_{\mathrm{loc}}(B(U), B(U)) := \mathrm{Hom}_{\mathrm{loc}}((\mathbb{A}_{\mathrm{cpt}}(\pi^{-1}(U)) \rtimes \mathbb{Z}_2)^{\hat{\otimes} k}, \mathbb{A}(\pi^{-1}(U)) \rtimes \mathbb{Z}_2).$$

Here the subscript "loc" means that we only include those cochain Ψ of $\mathrm{Hom}((\mathbb{A}_{\mathrm{cpt}}(\pi^{-1}(U)) \rtimes\mathbb{Z}_2)^{\hat{\otimes} k}, \mathbb{A}(\pi^{-1}(U)) \rtimes \mathbb{Z}_2)$ such that

$$\pi(\operatorname{supp}\Psi(a_1, \ldots, a_k)) \subset \cap_{i=1}^{k}\pi(\operatorname{supp} a_i). \tag{27}$$

Let $\mathrm{C}^\bullet_{\mathrm{loc}}(B, B)$ denote the subcomplex of $\mathrm{C}^\bullet(B, B)$ comprising cochains satisfying the locality condition (27). By [19, Proposition 4.1], the natural map of complexes $\mathrm{C}^\bullet_{\mathrm{loc}}(B, B) \to \mathrm{C}^\bullet(B, B)$ is a quasi-isomorphism. Using a spectral sequence argument as well as a localization procedure similar to that in [15, 23], show that the natural maps $\mathrm{C}_\bullet(B) \to \mathcal{C}_{B,\bullet}(X)$ and $\mathrm{C}^\bullet_{\mathrm{loc}}(B, B) \to \mathcal{C}^\bullet_B(X)$ are quasi-isomorphisms. Further, note that a cap product with $\theta|_U$ gives a map of complexes from $\mathcal{C}^\bullet_B(U)$ to $\mathcal{C}_{B,2n-\bullet}(U)$ for each open $U \subset X$. To see why this is the case, observe that even though $\theta|_U$ is not compactly supported, its cap product with an element Ψ of $\mathrm{C}^k_{\mathrm{loc}}(B(U), B(U))$ is well defined. This is because $\Psi(a_1, \ldots, a_k)$ makes sense even if $a_1 \otimes \cdots \otimes a_k$ is not compactly supported: the condition (27) is equivalent to the condition that, for any x in $\pi^{-1}(U)$, $\psi(a_1, \ldots, a_k)(x)$ depends only on the jets of $a_1, \ldots, a_k$ at x and at $\gamma.x$ (see, for instance, the proof of [19, Proposition 4.11]). It follows that $\theta \cap -$ is defined as a map of complexes of sheaves from $\mathcal{C}^\bullet_B$ to $\mathcal{C}_{B,2n-\bullet}$. Hence, one has the following commutative diagram:

$$\begin{tikzcd}
\mathrm{C}^\bullet(B, B) \arrow[r, "\theta\cap-"] & \mathrm{C}_{2n-\bullet}(B) \\
\mathrm{C}^\bullet_{\mathrm{loc}}(B, B) \arrow[u] \arrow[r, "\theta\cap-"] \arrow[d] & \mathrm{C}_{2n-\bullet}(B) \arrow[u, "\mathrm{id}"'] \arrow[d] \\
\mathcal{C}^\bullet_B(X) \arrow[r, "\theta\cap-"'] & \mathcal{C}_{B,2n-\bullet}(X)
\end{tikzcd}.$$

Since the vertical arrows in the above diagram are quasi-isomorphisms, it suffices to check that the bottom arrow is a quasi-isomorphism. Since $\mathcal{C}_{B,\bullet}$ as well as $\mathcal{C}^\bullet_B$ are complexes of fine sheaves on X, the bottom arrow in the above diagram is the map induced by $\theta \cap -$ on hypercohomologies. It therefore suffices to verify that $\theta \cap - : \mathcal{C}^\bullet_B \to \mathcal{C}_{B,2n-\bullet}$ is a quasi-isomorphism of complexes of sheaves on X. This is a local verification. We outline it for a sufficiently small Darboux neighborhood $U := V/\mathbb{Z}_2$ of a point on the singular locus of X. The (more straightforward) verification for U, a neighborhood of a point not contained in the singular locus, is left to the reader.

Step 2: Let V be have Darboux coordinates $p_1, q_1, \ldots, p_n, q_n$ with $x_{2i-1} := p_i$ and $x_{2i} := q_i$. By [9, Theorem 5.5.1], we may assume that $\mathbb{A}_V$ is isomorphic to a quantum algebra with trivial Fedosov connection. Note that $\theta|_U$ is a Hochschild $2n$-cycle of $\mathbb{A}_V \rtimes \mathbb{Z}_2$ such that $\chi_0(\theta|_U) = 1$. We need to check that $\theta|_U \cap -$ is a quasi-isomorphism. Recall that $\mathrm{HH}_\bullet(\mathbb{A}_V \rtimes \mathbb{Z}_2) = (\mathrm{HH}_\bullet(\mathbb{A}_V) \oplus \mathrm{HH}_\bullet(\mathbb{A}_V, \mathbb{A}_{V,\gamma}))^{\mathbb{Z}_2}$. Consider the $\mathbb{Z}_2$-invariant (reduced) cycle

$$c_{2n} := 1 \otimes \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(2n)}.$$

Further recall that

$$\mathrm{HH}^\bullet_{\mathrm{loc}}(\mathbb{A}_{\mathrm{cpt},V} \rtimes \mathbb{Z}_2, \mathbb{A}_V \rtimes \mathbb{Z}_2) = (\mathrm{HH}^\bullet_{\mathrm{loc}}(\mathbb{A}_{\mathrm{cpt},V}, \mathbb{A}_V) \oplus \mathrm{HH}^\bullet_{\mathrm{loc}}(\mathbb{A}_{\mathrm{cpt},V}, \mathbb{A}_{V,\gamma}))^{\mathbb{Z}_2}.$$

Since $\chi_0(\theta|_U) = 1$ and $\chi_{\mathrm{FFS}}(c_{2n}) = 1$, our verification will be complete once we check that $c_{2n} \cap - : \mathrm{C}^\bullet_{\mathrm{loc}}(\mathbb{A}_{\mathrm{cpt},V}, \mathbb{A}_V) \to \mathrm{C}_{2n-\bullet}(\mathbb{A}_V)$ and $c_{2n} \cap - : \mathrm{C}^\bullet_{\mathrm{loc}}(\mathbb{A}_{\mathrm{cpt},V}, \mathbb{A}_{V,\gamma}) \to \mathrm{C}_{2n-\bullet}(\mathbb{A}_V, \mathbb{A}_{V,\gamma})$ are quasi-isomorphisms. $c_{2n} \cap - : \mathrm{C}^\bullet_{\mathrm{loc}}(\mathbb{A}_{\mathrm{cpt},V}, \mathbb{A}_V) \to \mathrm{C}_{2n-\bullet}(\mathbb{A}_V)$ is indeed a quasi-isomorphism: the only nontrivial cohomology on the left-hand side is generated by the 0-cocycle 1, and $c_{2n} \cap 1 = c_{2n}$.

It remains to check that $c_{2n} \cap - : \mathrm{C}^\bullet_{\mathrm{loc}}(\mathbb{A}_{\mathrm{cpt},V}, \mathbb{A}_{V,\gamma}) \to \mathrm{C}_{2n-\bullet}(\mathbb{A}_V, \mathbb{A}_{V,\gamma})$ is a quasi-isomorphism. For this, we assume without loss of generality that γ fixes $x_1, \ldots, x_{2n-2l}$. As in the proof of [19, Proposition 4.4], one still has a Koszul resolution of $\mathbb{A}_{\mathrm{cpt},V}$ as $\mathbb{A}_{\mathrm{cpt},V} \otimes \mathbb{A}^{\mathrm{op}}_{\mathrm{cpt},V}$ -bimodules. Using this resolution, $\mathrm{HH}^\bullet(\mathbb{A}_{\mathrm{cpt},V}, \mathbb{A}_{V,\gamma})$ may be calculated as the cohomology of the complex

$$K^p_\gamma := \wedge^p W \otimes \mathbb{A}_V,$$

with

$$d_\gamma(a \otimes \partial y_{i_1} \wedge \cdots \wedge \partial y_{i_p}) := \sum_{j=1}^{2n} (-1)^j (y_j \star a - a \star_\gamma y_j) \partial y_j \wedge \partial y_{i_1} \wedge \cdots \wedge \partial y_{i_p}.$$

Here, $y_i := x_{2i+1}$ for $1 \le i \le n$ and $y_i := x_{2(i-n)}$ for $i = n+1, \ldots, 2n$. W denotes the linear span of the ∂y_i's. Further, one has a projection from the (reduced) cochain complex $\mathrm{C}^\bullet_{\mathrm{loc}}(\mathbb{A}_V, \mathbb{A}_{V,\gamma})$ to $K^\bullet_\gamma$. Moreover, there exists a generator Ψ_γ of the unique nonzero cohomology of $\mathrm{C}^\bullet_{\mathrm{loc}}(\mathbb{A}_V, \mathbb{A}_{V,\gamma})$ whose restriction to $\wedge^{2l} W^*$ coincides with $\partial y_{2n-2l+1} \wedge \cdots \wedge \partial y_{2n}$. Now, c_{2n} may be viewed as the shuffle product of c_{2n-2l} formed out of $x_1, \ldots, x_{2n-2l}$ and c_{2l} formed out of $x_{2n-2l+1}, \ldots, x_{2n}$. This makes it easy to see that

$$c_{2n} \cap \Psi_\gamma = c_{2n-2l},$$

where the right-hand side of the above equation is formed out of $x_1, \ldots, x_{2n-2l}$. The latter is the generator of the (unique) nonzero homology $\mathrm{HH}_{2n-2l}(\mathbb{A}_V, \mathbb{A}_{V,\gamma})$. ■

Theorem 5.

$$\mathrm{HH}^\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))) \cong \mathrm{H}^\bullet_{\mathrm{CR}}(M/\mathbb{Z}_2, \mathbb{C})((\hbar_1))((\hbar_2)).$$ □

Proof. Let $A_0 := \mathbb{A}_{M/\mathbb{Z}_2}((\hbar_1))$. Let $P := \mathbf{e}B$ and let $Q := B\mathbf{e}$ where $\mathbf{e} := \frac{1+\gamma}{2}$. There are natural bimodule isomorphisms $u: P \otimes_B Q \to A_0$ and $v: Q \otimes_{A_0} P \to B$. One checks that (A_0, B, P, Q, u, v) forms a Morita context in the sense of [19, Appendix A7]. It follows that A_0 and B are Morita equivalent as bornological algebras.

Since $\mathrm{HH}_{2n}(B) \cong \mathbb{C}((\hbar_1))$, Proposition 13 can be restated to say that the map

$$\begin{aligned} \mathrm{HH}_{2n}(B) \otimes_{\mathbb{C}((\hbar_1))} \mathrm{HH}^\bullet(B, B) &\to \mathrm{HH}_{2n-\bullet}(B), \\ \alpha \otimes \beta &\mapsto \alpha \cap \beta, \end{aligned} \tag{28}$$

of $\mathbb{C}((\hbar_1))$-modules is an isomorphism. Since A_0 and B are Morita equivalent as bornological algebras, the map

$$\begin{aligned} \mathrm{HH}_{2n}(A_0) \otimes_{\mathbb{C}((\hbar_1))} \mathrm{HH}^\bullet(A_0, A_0) &\to \mathrm{HH}_{2n-\bullet}(A_0) \\ \alpha \otimes \beta &\mapsto \alpha \cap \beta, \end{aligned} \tag{29}$$

is an isomorphism of $\mathbb{C}((\hbar_1))$-modules.

Note that $\mathrm{C}_\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2)))$ as well as $\mathrm{C}^\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2)), \mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2)))$ are complexes filtered by powers of $\hbar_2$. By Theorem 4 and $\mathbb{C}((\hbar_1))((\hbar_2))$-linearity, there exists a $2n$-cycle θ in $\mathrm{C}_\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))[[\hbar_2]])$ representing a nonzero homology class in $\mathrm{HH}_{2n}(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2)))$. We claim that, for a suitable choice of θ, $\theta \cap -$ is an isomorphism. Let θ_0 denote the image of θ under the map of chain complexes induced by the homomorphism $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))[[\hbar_2]]$ mapping $\hbar_2$ to 0. Note that $\theta \cap -$ is a map of filtered complexes whose associated graded map is $\theta_0 \cap -$. Since the map (29) is an isomorphism, $\theta_0 \cap -$ is an isomorphism whenever $[\theta_0] \neq 0$. We, therefore, need to show the existence of a $2n$-cycle θ of $\mathrm{C}_\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))[[\hbar_2]])$ such that $[\theta]$ is nonzero in $\mathrm{HH}_{2n}(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2)))$ and $[\theta_0] \neq 0$. Proposition 17 in Section 4.5 does exactly this, proving the desired theorem. ■

4 Traces

As in Section 2.4, $\mathbb{K} := \mathbb{C}((\hbar_1))((\hbar_2))$. We construct a trace density

$$\chi^\gamma : \mathrm{C}_\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))) \to \Omega^{2n-2-\bullet}(M_2^\gamma)((\hbar_1))((\hbar_2)).$$

This is a map of complexes of $\mathbb{K}$-vector spaces (with the right-hand side equipped with the de-Rham differential). We prove an algebraic index theorem computing $\chi^\gamma(1)$. In fact, our construction of the trace density χ^γ extends to yield a map of complexes of presheaves $\mathrm{C}_\bullet(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2))) \to i_{2*}\Omega^{2n-2-\bullet}_{M_2^\gamma}((\hbar_1))((\hbar_2))$ on M. This may be viewed as a map $\lambda : \mathrm{C}_\bullet(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2))) \to i_{2*}\underline{\mathbb{K}}_{M_2^\gamma}[2n-2]$ in the derived category of presheaves on M. Similar trace densities $\lambda_l : \mathrm{C}_\bullet(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2))) \to i_{2l*}\underline{\mathbb{K}}_{M_{2l}^\gamma}[2n-2l]((\hbar_1))((\hbar_2))$ can be constructed for $l > 1$. The key parts of the latter construction are in [17]. We skip the latter construction in order to avoid being repetitive. We also construct a map $\mu : \mathrm{C}_\bullet(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2))) \to \underline{\mathbb{K}}_M[2n]$ in the derived category of presheaves on M. At the end of this section, we will show that $\lambda \oplus \mu$ is a quasi-isomorphism when $M^\gamma = M_2^\gamma$. More generally, the same method (with some more notation chasing) as that used at the end of this section shows that $\oplus_{l>1}\lambda_l \oplus \lambda \oplus \mu$ is a quasi-isomorphism. This completes the proof of Theorem 4. The map μ mentioned above yields a trace χ_0 on $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ such that χ_0 and χ^γ are linearly independent over $\mathbb{C}((\hbar_1))((\hbar_2))$. In Section 4.5, we discuss some basic properties of χ_0. In the process, we obtain a proposition that completes the proof of Theorem 5.

4.1 Preliminaries

Notations in this section are as in Section 3.1.1. Recall that a sheaf $\mathcal{A}_D$ of algebras on M_2^γ is constructed by constructing a Fedosov connection D on $\wedge^\bullet T^*M_2^\gamma \otimes \mathcal{W} \otimes_{\mathbb{C}((\hbar_1))} \mathcal{V}$. In [11], D was constructed starting with a symplectic connection ∇_T on TM_2^γ and a Hermitian connection ∇_N on N. These induce connections ∂_T and ∂_N on $\mathcal{W}$ and $\mathcal{V}$, respectively. This makes $\nabla := \partial_T \otimes 1 + 1 \otimes \partial_N$ a connection on $\mathcal{W} \otimes_{\mathbb{C}((\hbar_1))} \mathcal{V}$. ∇ automatically extends to a connection on $\bigwedge^\bullet T^*M_2^\gamma \otimes \mathcal{W} \otimes_{\mathbb{C}((\hbar_1))} \mathcal{V}$. The connection D is then constructed as a sum

$$D = \nabla + [A, -]_{\hbar_1}, \tag{30}$$

where $A \in \Omega^1(M_2^\gamma, \mathcal{W} \otimes_{\mathbb{C}((\hbar_1))} \mathcal{V})$. Recall that $D^2 = [\Theta, -]_{\hbar_1}$ where $\Theta \in \Omega^2(M_2^\gamma, \mathcal{W} \otimes_{\mathbb{C}((\hbar_1))} \mathcal{V})$ is central. This implies that $\Theta \in \Omega^2(M_2^\gamma, \mathbb{K})$.

Recall from Equation (12) that one has a map of Lie algebras $\mathfrak{sp}_{2n-2}(\mathbb{K}) \oplus \mathfrak{u}_1(\mathbb{K}) \to \mathbb{W}_{n-1}((\hbar_1)) \otimes_{\mathbb{C}((\hbar_1))} \mathbb{D}_2((\hbar_1))((\hbar_2))$. This induces a map $\mathfrak{sp}(TM_2^\gamma) \oplus \mathfrak{u}(N) \to \mathcal{W} \otimes_{\mathbb{C}((\hbar_1))} \mathcal{V}$ of bundles of Lie algebras over M_2^γ. We recall from [11] that

$$\nabla^2 = [R_T + R_N, -]_{\hbar_1}, \tag{31}$$

where $R_T \in \Omega^2(M_2^\gamma, \mathfrak{sp}(TM_2^\gamma))$ and $R_N \in \Omega^2(M_2^\gamma, \mathfrak{u}(N))$. It follows that

$$\nabla A + \tfrac{1}{2}[A, A]_{\hbar_1} = \Theta - R_T - R_N. \tag{32}$$

We remark that changing ∇_T and ∇_N changes ∇ (and hence A) by an element of $\Omega^1(M_2^\gamma, \mathfrak{sp}(TM_2^\gamma) \oplus \mathfrak{u}(N))$.

4.2 The first trace: construction of χ^γ

The construction of χ^γ follows the trace density construction from [5]. Since this construction has been done in detail even in subsequent works (see [18, 20, 21, 26] for analogs of this construction for cyclic and Lie chains, respectively), we just specify the steps in this construction without providing detailed proofs of their validity.

Let U be an open subset of M_2^γ on which TM_2^γ and N are trivial. Choosing trivializations of TM_2^γ and N over U, one obtains an identification of sheaves of DG-algebras

$$\left(\bigwedge^{\bullet} T^*M_2^\gamma \otimes \mathcal{W} \otimes_{\mathbb{C}((\hbar_1))} \mathcal{V}, D\right) \cong (\Omega^\bullet(U, \mathbb{W}_{n-1}((\hbar_1)) \otimes_{\mathbb{C}((\hbar_1))} \mathbb{D}_2((\hbar_1))((\hbar_2))), d + [\theta, -]_{\hbar_1}), \tag{33}$$

where $\theta \in \Omega^1(U, \mathbb{W}_{n-1}((\hbar_1)) \otimes_{\mathbb{C}((\hbar_1))} \mathbb{D}_2((\hbar_1))((\hbar_2)))$ satisfies the Maurer–Cartan condition

$$d\theta + \tfrac{1}{2}[\theta, \theta]_{\hbar_1} \in \Omega^2(U, \mathbb{K}).$$

Recall that $\mathcal{A}_D$ is the sheaf of degree 0 flat sections of $(\bigwedge^\bullet T^*M_2^\gamma \otimes \mathcal{W} \otimes_{\mathbb{C}((\hbar_1))} \mathcal{V}, D)$. Let $\times$ denote the shuffle product on Hochschild chains and let $(\theta)^k$ denote the Hochschild chain $1 \otimes \theta \otimes \cdots \otimes \theta$ with k factors θ. Following [5], one constructs a map

$$\begin{aligned} \chi^\gamma : \mathrm{C}_\bullet(\Gamma(U, \mathcal{A}_D)) &\to \Omega_U^{2n-2-\bullet}((\hbar_1, \hbar_2)) \\ a &\mapsto \sum_{k=0}^{\infty} (-1)^{\lfloor \frac{k}{2} \rfloor} \psi_{2n-2}(a \times (\theta)^k), \end{aligned} \tag{34}$$

of complexes of sheaves on U. We remark that, in [5], $d\theta + \frac{1}{2}[\theta, \theta] = 0$. However, as pointed out in [18], Equation (34) gives a map of complexes in our situation as well. Moreover, a different choice of trivialization of TM_2^γ and of trivialization of N changes θ by an element of $\Omega^1(U, \mathfrak{sp}_{2n-2}(\mathbb{K}) \oplus \mathfrak{u}_1(\mathbb{K}))$. Since the cocycle ψ_{2n-2} is $\mathfrak{sp}_{2n-2}(\mathbb{K}) \oplus \mathfrak{u}_1(\mathbb{K})$-basic (see Proposition 4), this change leaves the map χ^γ unchanged. As a result, the map χ^γ is well defined globally on M_2^γ, giving us a map of complexes

$$\chi^\gamma : \mathrm{C}_\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))) \to \Omega^{2n-2-\bullet}(M_2^\gamma)((\hbar_1))((\hbar_2)),$$

(note that there is a natural map of algebras $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2)) \to \mathcal{A}_D$). Also, after a choice of trivialization of TM_2^γ and of N over U, A differs from θ by an element of $\Omega^1(U, \mathfrak{sp}_{2n-2}(\mathbb{K}) \oplus \mathfrak{u}_1(\mathbb{K}))$. It follows that Equation (34) may be rewritten as

$$a \mapsto \sum_{k=0}^{\infty} (-1)^{\lfloor \frac{k}{2} \rfloor} \psi_{2n-2}(a \times (A)^k). \tag{35}$$

4.2.1 *A basic example*

Let us consider the case when $M \subset \mathbb{R}^{2n}$ is an open neighborhood of the origin with symplectic form

$$\omega = \sum_{i=1}^{n-1} dp_i \wedge dq_i + \frac{1}{2i} dz \wedge d\bar{z}.$$

Let $\mathbb{Z}_2$ act on M such that $(p, q, z, \bar{z}) \mapsto (p, q, -z, -\bar{z})$. Then M_2^γ is given by the equation $z = \bar{z} = 0$ with $p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1}$ being Darboux coordinates for M_2^γ. In this case, the connection $\nabla := \partial_T \otimes 1 + 1 \otimes \partial_N$ (see Section 4.1) may be chosen to be d itself. Then, $D := d + \delta$ is itself a (trivial) Fedosov connection where $\delta = \sum_{i=1}^{2n-2} dx^i \frac{\partial}{\partial y^i}$. We remark that here $x^{2i-1} = p_i$ and $x^{2i} = q_i$ with y^i being the corresponding fiberwise coordinates. For this trivial Fedosov connection, the form θ is given by $\sum_{1 \le i, j \le 2n-2} \omega_{ij} y^i \, dx^j$ where $\omega_{ij} = 1$ if $\{i, j\} = \{2k-1, 2k\}$ for some k and $\omega_{ij} = 0$ otherwise. Therefore

$$(\theta)^{2n-2} = \sum_{\sigma \in S_{2n-2}} \mathrm{sgn}(\sigma) 1 \otimes y^{\sigma(1)} \otimes \cdots \otimes y^{\sigma(2n-2)} dp_1 \wedge dq_1 \wedge \cdots \wedge dp_{n-1} \wedge dq_{n-1}.$$

For any function on M of the form $F = f(p, q).g(z, \bar{z})$, we see that

$$\chi^\gamma(F) = \psi_{2n-2}(F \times (\theta)^{2n-2}) = \phi(g).\tau_{2n-2}(f \times (\theta)^{2n-2})$$
$$= \phi(g) f dp_1 \wedge dq_1 \wedge \cdots \wedge dp_{n-1} \wedge dq_{n-1}.$$

Here $\phi(g)$ is given by the formula in Theorem 2. We may therefore rewrite the above local formula as

$$\chi^\gamma(F) = \sum_{k=0}^{\infty} \frac{(i\hbar_1)^k}{(k!)^2} \cdot \left(\prod_{l=1}^{k} \left\{ \frac{l}{2} + (-1)^{l+1} \frac{2\lfloor \frac{l+1}{2} \rfloor \hbar_2}{l+1} \right\} \right) \cdot \left. \frac{\partial^{2k} F}{\partial z^k \partial \bar{z}^k} \right|_{z=\bar{z}=0} \tag{36}$$
$$dp_1 \wedge dq_1 \wedge \cdots \wedge dp_{n-1} \wedge dq_{n-1}.$$

One can then argue as in the end of the proof of Theorem 2 to show that Equation (36) is valid for arbitrary functions on M as well. Another point to note is that

$$\chi^\gamma \left(\sum_{\sigma \in S_{2n-2}} \mathrm{sgn}(\sigma) 1 \otimes y^{\sigma(1)} \otimes \cdots \otimes y^{\sigma(2n-2)} \right) = 1. \tag{37}$$

4.2.2 *A remark for general M*

We return to the general case, as in the beginning of this section. We recall from [9] (see Section 5) that, for every $x \in M_2^\gamma$, the symplectic form ω on M may be identified with $\sum_{i=1}^{n-1} dp_i \wedge dq_i + \frac{1}{2i} dz \wedge d\bar{z}$ over $V := N_{\alpha,U}$ for sufficiently small α for some neighborhood U of x in M_2^γ. We denote the trivial Fedosov connection in Example 4.2.1 by D^0. We also denote the corresponding sheaf of algebras on U by $\mathcal{A}_{D^0}$. Further, by imitating the proof of [8, Theorem 5.5.1], one can prove the following analog of the "quantum Darboux theorem" [8, Theorem 5.5.1].

Proposition 14. $\Gamma(W, \mathcal{A}_D)$ is locally isomorphic to $\Gamma(W, \mathcal{A}_{D^0})$ for some neighborhood W of x contained in U. □

4.3 The algebraic index theorem

Let Θ, R_T, R_N be as in Section 4.1.

Theorem 6. The $2n-2$-form

$$\chi^\gamma(1) - \hbar_1^{n-1}\left(\hat{A}(R_T)\mathrm{Ch}\left(-\frac{\Theta}{\hbar_1}\right)\mathrm{Ch}_\phi\left(\frac{R_N}{\hbar_1}\right)\right)_{n-1},$$

is an exact form on M_2^γ. □

Proof. Let $P = (\hat{A}_{\hbar_1}\mathrm{Ch}\mathrm{Ch}_\phi)_{n-1}$. Then

$$\begin{aligned}\chi^\gamma(1) = (-1)^{n-1}\psi_{2n-2}((A)^{2n-2}) &= \frac{(-1)^{n-1}}{(2n-2)!}(\mathrm{ev}_1\Psi_{2n-2})(A^{2n-2})\\ &= (-1)^{n-1}(\mathrm{ev}_1\Psi_{2n-2})(A\wedge\cdots\wedge A).\end{aligned}$$

By Theorem 3, it follows that, modulo exact forms,

$$\chi^\gamma(1) = \chi(P)(A\wedge\cdots\wedge A).$$

Hence, for any vector fields $\xi_1,\ldots,\xi_{2n-2}$ on M_2^γ

$$\begin{aligned}\chi^\gamma(1)(\xi_1,\ldots,\xi_{2n-2}) &= \chi(P)(A(\xi_1)\wedge\cdots\wedge A(\xi_{2n-2}))\\ &= \frac{1}{(n-1)!}\sum_{\sigma(2i-1)<\sigma(2i)}(-1)^\sigma P(C(A(\xi_{\sigma(1)}),A(\xi_{\sigma(2)})),\ldots,\\ &\quad C(A_{\sigma(2n-3)},A_{\sigma(2n-2)})).\end{aligned}$$

As in [10], A may be chosen so that $\mathrm{pr}(A)=0$. In this case, for smooth vector fields ξ,η on M,

$$C(A(\xi),A(\eta)) = -\mathrm{pr}[A(\xi),A(\eta)]_{\hbar_1} = -\mathrm{pr}(\nabla A(\xi,\eta)+[A(\xi),A(\eta)]_{\hbar_1}) = (\Theta - R_T - R_N)(\xi,\eta).$$

The last equality above is due to (32). It follows that

$$\begin{aligned}\chi^\gamma(1)(\xi_1,\ldots,\xi_{2n-2}) = \frac{1}{(n-1)!}\sum_{\sigma(2i-1)<\sigma(2i)}(-1)^\sigma P(C(A(\xi_{\sigma(1)}),A(\xi_{\sigma(2)})),\ldots,\\ C(A_{\sigma(2n-3)},A_{\sigma(2n-2)}))\end{aligned}$$

$$= \frac{1}{(n-1)!} \sum_{\sigma(2i-1)<\sigma(2i)} (-1)^{\sigma} P((\Theta - R_T - R_N)(\xi_{\sigma(1)}, \xi_{\sigma(2)}), \dots,$$

$$(\Theta - R_T - R_N)(\xi_{\sigma(2n-3)}, \xi_{\sigma(2n-2)}))$$

$$= \frac{1}{(n-1)!} P((\Theta - R_T - R_N)^{n-1})(\xi_1, \dots, \xi_{2n-2}).$$

Recall that, for any $X \in \mathfrak{sp}_{2n-2}(\mathbb{K}) \oplus \mathfrak{gl}_N(\mathbb{K}) \oplus \mathfrak{u}_1(\mathbb{K})$,

$$\frac{1}{(n-1)!} P(X, \dots, X) = P(X),$$

where the element P of $(S^{n-1}\mathfrak{h}^*)^{\mathfrak{h}}$ is being viewed as a linear form $\mathfrak{h}^{\otimes n-1}$ in the left-hand side and a polynomial function on $\mathfrak{h}$ in the right-hand side of the above equation. Hence, modulo exact forms,

$$\chi^{\gamma}(1) = (\hat{A}_{\hbar_1}(R_T)\mathrm{Ch}(-\Theta)\mathrm{Ch}_{\phi}(R_N))_{n-1} = \hbar_1^{n-1}\left(\hat{A}(R_T)\mathrm{Ch}\left(-\frac{\Theta}{\hbar_1}\right)\mathrm{Ch}_{\phi}\left(\frac{R_N}{\hbar_1}\right)\right)_{n-1}.$$

This completes the proof of the desired result. ■

4.4 Proof of Proposition 12

We will prove the proposition for the case when $M^{\gamma} = M_2^{\gamma}$. The general case is proved in a completely similar fashion (with some extra notation to keep track of). For this case, we denote i_2 by i. Let $j : M^- = M \setminus M^{\gamma} \to M$ denote the natural inclusion. Let $\mathrm{C}^{\mathrm{sh}}_{\bullet}(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2)))$ denote the sheafification of the complex of presheaves $\mathrm{C}_{\bullet}(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2)))$. We note that $j^*\mathrm{C}^{\mathrm{sh}}_{\bullet}(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2))) = \mathrm{C}^{\mathrm{sh}}_{\bullet}(\mathcal{A}_{M^-}((\hbar_1))((\hbar_2)))$, where $\mathcal{A}_{M^-}((\hbar_1))((\hbar_2))$ is the Fedosov quantization of the sheaf of smooth functions on M^- with Weyl curvature ω. One can apply the trace density construction of [5] to extend the construction in [10] to a map of complexes $\chi_{\mathrm{FFS}} : \mathrm{C}^{\mathrm{sh}}_{\bullet}(\mathcal{A}_{M^-}((\hbar_1))((\hbar_2))) \to \Omega^{2n-\bullet}_{M^-}((\hbar_1))((\hbar_2))$. One has the following composite map in the derived category $\mathrm{D}(\mathrm{Sh}_{\mathbb{K}}(M))$ of sheaves of $\mathbb{K}$-vector spaces on M, which we shall denote by μ^{sh}:

$$\mathrm{C}^{\mathrm{sh}}_{\bullet}(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2))) \to Rj_*\mathrm{C}^{\mathrm{sh}}_{\bullet}(\mathcal{A}_{M^-}((\hbar_1))((\hbar_2))) \xrightarrow{Rj_*\chi_{\mathrm{FFS}}} Rj_*\Omega^{2n-\bullet}_{M^-}((\hbar_1))((\hbar_2))$$

$$Rj_*\Omega^{2n-\bullet}_{M^-}((\hbar_1))((\hbar_2)) \cong Rj_*\underline{\mathbb{K}}_{M^-}[2n] \cong \underline{\mathbb{K}}[2n].$$

The last isomorphism in $\mathrm{D}(\mathrm{Sh}_{\mathbb{K}}(M))$ is because $\underline{\mathbb{K}}$ is an injective sheaf on M^- (which implies that $Rj_*\underline{\mathbb{K}}_{M^-} \cong R^0 j_*\underline{\mathbb{K}}_{M^-}$) and also because M_2^γ is of codimension 2 (which implies that $R^0 j_*\underline{\mathbb{K}}_{M^-} \cong \underline{\mathbb{K}}$). As a result, in the derived category $\mathrm{D}(\mathrm{PrSh}_{\mathbb{K}}(M))$ of presheaves of $\mathbb{K}$-vector spaces on M, one obtains the following composite map, which shall be denoted by μ:

$$\mathrm{C}_\bullet(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2))) \to \mathrm{C}_\bullet^{\mathrm{sh}}(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2))) \to \underline{\mathbb{K}}[2n].$$

Here, the first arrow is the sheafification and the second arrow is the forgetful functor applied to μ^{sh}. Proposition 12 is implied by the following stronger proposition.

Proposition 15. $\mu \oplus \lambda : \mathrm{C}_\bullet(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2))) \to \underline{\mathbb{K}}[2n] \oplus i_*\underline{\mathbb{K}}_{M_2^\gamma}[2n-2]$ is an isomorphism in $\mathrm{D}(\mathrm{PrSh}_{\mathbb{K}}(M))$. □

Proof. It suffices to verify that, for sufficiently small open subsets V of M, $(\mu \oplus \lambda)|_V$ induces an isomorphism on homologies. We shall do this case by case.

Case 1: $V \subset M^-$. In this case, $i_*\underline{\mathbb{K}}_{M_2^\gamma}|_V = 0$. We therefore need to verify that $\mu|_V : \mathrm{C}_\bullet(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2)))|_V \to \underline{\mathbb{K}}[2n]|_V$ induces an isomorphism on homologies. But the restriction $|_V$ factors through j^*. Hence $\mu|_V$ coincides with the composite map

$$\mathrm{C}_\bullet(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2)))|_V \to \mathrm{C}_\bullet(\mathcal{A}_{M^-}((\hbar_1))((\hbar_2)))|_V \stackrel{\chi_{\mathrm{FFS}}}{\to} \Omega_{M^-}^{2n-\bullet}((\hbar_1))((\hbar_2))|_V,$$

in $\mathrm{D}(\mathrm{PrSh}_{\mathbb{K}}(V))$. The first arrow in the above composition is a term by term isomorphism of complexes of presheaves on V. The second arrow is known to be a quasi-isomorphism. The reader may see [5] for the analogous assertion for the Hochschild chain complex of the sheaf of holomorphic differential operators on a compact, complex manifold. Also, a cyclic homology analog of this assertion is available in [26, Section 5.2].

Case 2: $V = N_{\alpha,U}$ for a sufficiently small open subset U of M_2^γ. This case is a little more involved. By Proposition 2, $\mathrm{C}_\bullet(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2)))|_V$ is quasi-isomorphic to $\underline{\mathbb{K}}_V[2n] \oplus i_*\underline{\mathbb{K}}_U[2n-2]$ as complexes of presheaves on V. Since the sheafification is exact, $\mathrm{C}_\bullet^{\mathrm{sh}}(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2)))|_V$ is quasi-isomorphic to $\underline{\mathbb{K}}_V[2n] \oplus i_*\underline{\mathbb{K}}_U[2n-2]$ with the natural adjunction from $\mathrm{C}_\bullet(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2)))|_V$ to $\mathrm{C}_\bullet^{\mathrm{sh}}(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2)))|_V$ inducing the identity on homologies. For the rest of this proof, we identify $\mathrm{C}_\bullet(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2)))|_V$ as well as its sheafification with $\underline{\mathbb{K}}_V[2n] \oplus i_*\underline{\mathbb{K}}_U[2n-2]$ wherever it is convenient for us to do so. With this identification, one may view $\mu^{\mathrm{sh}}|_V$ as a map in $\mathrm{D}(\mathrm{Sh}_{\mathbb{K}}(V))$ from $\underline{\mathbb{K}}_V[2n] \oplus i_*\underline{\mathbb{K}}_U[2n-2]$

to $\underline{\mathbb{K}}_V[2n]$. Note that

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}_{\mathbb{K}}(V))}(i_*\underline{\mathbb{K}}_U[2n-2], \underline{\mathbb{K}}_V[2n]) &= \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}_{\mathbb{K}}(V))}(i_*\underline{\mathbb{K}}_U[2n-2], j_*\underline{\mathbb{K}}_{V\setminus U}[2n]) \\ &= \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}_{\mathbb{K}}(V\setminus U))}(j^*i_*\underline{\mathbb{K}}_U[2n-2], \underline{\mathbb{K}}_{V\setminus U}[2n]) = 0.\end{aligned}$$

It follows that the composite map

$$i_*\underline{\mathbb{K}}_U[2n-2] \to i_*\underline{\mathbb{K}}_U[2n-2] \oplus \underline{\mathbb{K}}_V[2n] \xrightarrow{\mu^{\mathrm{sh}}} \underline{\mathbb{K}}_V[2n],$$

is zero in $\mathrm{D}(\mathrm{Sh}_{\mathbb{K}}(V))$. On the other hand, the composite map

$$\underline{\mathbb{K}}_V[2n] \to i_*\underline{\mathbb{K}}_U[2n-2] \oplus \underline{\mathbb{K}}_V[2n] \xrightarrow{\mu^{\mathrm{sh}}} \underline{\mathbb{K}}_V[2n],$$

is nonzero. This is because, applying j^* to the above composite gives the composite map

$$\underline{\mathbb{K}}_{V\setminus U}[2n] \to \mathrm{C}^{\mathrm{sh}}_\bullet(\mathcal{A}_{M^-}((\hbar_1))((\hbar_2)))|_{V\setminus U} \xrightarrow{\chi_{\mathrm{FFS}}} \Omega^{2n-\bullet}_{M^-}((\hbar_1))((\hbar_2))|_{V\setminus U} \cong \underline{\mathbb{K}}_{V\setminus U}[2n],$$

which is the identity if the first quasi-isomorphism above is suitably chosen. It follows that the composite map

$$\underline{\mathbb{K}}_V[2n] \to i_*\underline{\mathbb{K}}_U[2n-2] \oplus \underline{\mathbb{K}}_V[2n] \xrightarrow{\mu} \underline{\mathbb{K}}_V[2n],$$

is an isomorphism in $\mathrm{D}(\mathrm{PrSh}_{\mathbb{K}}(V))$ and the composite map

$$i_*\underline{\mathbb{K}}_U[2n-2] \to i_*\underline{\mathbb{K}}_U[2n-2] \oplus \underline{\mathbb{K}}_V[2n] \xrightarrow{\mu} \underline{\mathbb{K}}_V[2n],$$

is zero in $\mathrm{D}(\mathrm{PrSh}_{\mathbb{K}}(V))$.

The remaining task is to check that the composite map

$$\underline{\mathbb{K}}_V[2n] \to i_*\underline{\mathbb{K}}_U[2n-2] \oplus \underline{\mathbb{K}}_V[2n] \xrightarrow{\lambda} i_*\underline{\mathbb{K}}_U[2n-2],$$

is zero and that the composite map

$$i_*\underline{\mathbb{K}}_U[2n-2] \to i_*\underline{\mathbb{K}}_U[2n-2] \oplus \underline{\mathbb{K}}_V[2n] \xrightarrow{\lambda} i_*\underline{\mathbb{K}}_U[2n-2],$$

is an isomorphism in $\mathrm{D}(\mathrm{PrSh}_{\mathbb{K}}(V))$. We first note that, by its very construction, λ factors through $\mathrm{C}_\bullet^{\mathrm{sh}}(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2)))|_V$. We may therefore verify these facts in $\mathrm{D}(\mathrm{Sh}_{\mathbb{K}}(V))$. For the first of the above two facts, note that

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}_{\mathbb{K}}(V))}(\underline{\mathbb{K}}_V[2n], i_*\underline{\mathbb{K}}_U[2n-2]) = \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}_{\mathbb{K}}(U))}(\underline{\mathbb{K}}_U[2n], \underline{\mathbb{K}}_U[2n-2]) = 0.$$

The second fact is immediate from Proposition 14 and Equation (37) in the example provided in Section 4.2.1. ■

4.5 The other trace

As in the previous subsection, we assume that $M^\gamma = M_2^\gamma$ in order to avoid notational complexities. Our methods in this subsection work in the general situation with trivial modifications. The map

$$\mathbb{H}(\mu) : \mathbb{H}^{-\bullet}(M, \mathrm{C}_\bullet(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))((\hbar_2)))) \to \mathrm{H}^{2n-\bullet}(M, \mathbb{K}),$$

induced by μ on hypercohomologies is $\mathbb{Z}_2$-equivariant. In particular, by Proposition 10, it induces a map

$$\chi_0 : \mathrm{HH}_\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))) \to \mathrm{H}^{2n-\bullet}(M, \mathbb{K})^{\mathbb{Z}_2}.$$

The degree 0 component of this map $\chi_0 : \mathrm{HH}_0(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))) \to \mathrm{H}^{2n}(M, \mathbb{K})^{\mathbb{Z}_2}$ is another trace on $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$. Unfortunately, since the construction of χ_0 is not completely explicit, we are not in a position to prove an algebraic index theorem computing $\chi_0(1)$. It would indeed be interesting to have an explicit index formula computing $\chi_0(1)$. We however discuss some basic facts about χ_0 that enable us to prove a proposition completing the proof of Theorem 5.

We begin with the observation that μ restricts to a well-defined map

$$\mathrm{C}_\bullet(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))[\![\hbar_2]\!]) \xrightarrow{\mu} \underline{\mathbb{C}}((\hbar_1))[\![\hbar_2]\!][2n] \text{ in } \mathrm{D}(\mathrm{PrSh}_{\mathbb{C}((\hbar_1))[\![\hbar_2]\!]}(M)).$$

By Proposition 11, χ_0 "restricts" to a map of graded $\mathbb{C}((\hbar_1))[\![\hbar_2]\!]$-modules

$$\chi_0 : \mathrm{HH}_\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))[\![\hbar_2]\!]) \to \mathrm{H}^{2n-\bullet}(M, \mathbb{C}((\hbar_1))[\![\hbar_2]\!])^{\mathbb{Z}_2}.$$

One has an algebra homomorphism $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))[\![\hbar_2]\!] \to \mathbb{A}_{M/\mathbb{Z}_2}((\hbar_1))$ induced by the map $\mathbb{C}((\hbar_1))[\![\hbar_2]\!] \to \mathbb{C}((\hbar_1))$ taking $\hbar_2$ to 0. The following proposition asserts that χ_0 "quantizes" χ_{FFS}.

Proposition 16. The following diagram commutes:

$$\begin{array}{ccc} \mathrm{HH}_\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))[\![\hbar_2]\!]) & \xrightarrow{\hbar_2 \mapsto 0} & \mathrm{HH}_\bullet(\mathbb{A}_{M/\mathbb{Z}_2}((\hbar_1))) \\ \big\downarrow \chi_0 & & \chi_{\mathrm{FFS}} \big\downarrow \\ \mathrm{H}^{2n-\bullet}(M, \mathbb{C}((\hbar_1))[\![\hbar_2]\!])^{\mathbb{Z}_2} & \xrightarrow{\hbar_2 \mapsto 0} & \mathrm{H}^{2n-\bullet}(M, \mathbb{C}((\hbar_1)))^{\mathbb{Z}_2}. \end{array}$$

□

Proof. The construction of μ can be imitated step by step with $\mathcal{A}_{\mathrm{Dunkl}}$ replaced by $\mathcal{A}$ to obtain a map $\nu : \mathrm{C}_\bullet(\mathcal{A}((\hbar_1))) \to \underline{\mathbb{C}}((\hbar_1))[2n]$ in $\mathrm{D}(\mathrm{PrSh}_{\mathbb{C}((\hbar_1))}(M))$. One clearly has the following commutative diagram in $\mathrm{D}(\mathrm{PrSh}_{\mathbb{C}((\hbar_1))}(M))$:

$$\begin{array}{ccc} \mathrm{C}_\bullet(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))[\![\hbar_2]\!]) & \xrightarrow{\hbar_2 \mapsto 0} & \mathrm{C}_\bullet(\mathcal{A}((\hbar_1))) \\ \big\downarrow \mu & & \nu \big\downarrow \\ \underline{\mathbb{C}}((\hbar_1))[\![\hbar_2]\!][2n] & \xrightarrow{\hbar_2 \mapsto 0} & \underline{\mathbb{C}}((\hbar_1))[2n]. \end{array}$$

Further, one also has the following commutative diagram in $\mathrm{D}(\mathrm{Sh}_{\mathbb{C}((\hbar_1))}(M))$:

$$\begin{array}{ccc} \mathrm{C}_\bullet(\mathcal{A}((\hbar_1))) & \xrightarrow{\mathrm{id}} & \mathrm{C}_\bullet(\mathcal{A}((\hbar_1))) \\ \big\downarrow \chi_{\mathrm{FFS}} & & \nu^{\mathrm{sh}} \big\downarrow \\ \underline{\mathbb{C}}((\hbar_1))[2n] & \longrightarrow & Rj_*\underline{\mathbb{C}}((\hbar_1))[2n]. \end{array}$$

The bottom arrow in the above diagram is the natural adjunction map. The desired proposition follows from the two commutative diagrams described above once we observe that the map $\mathrm{H}^\bullet(M, \mathbb{C}) \to \mathbb{H}^\bullet(M, Rj_*\underline{\mathbb{C}}) \cong \mathrm{H}^\bullet(M, \mathbb{C})$ induced via the adjunction $\underline{\mathbb{C}} \to Rj_*\underline{\mathbb{C}} \cong \underline{\mathbb{C}}$ is the identity. ■

We remark that, with Proposition 16, one can see without difficulty that χ_0 and χ^γ are linearly independent over $\mathbb{C}((\hbar_1))((\hbar_2))$. Further, note that the argument in the proof of Proposition 15 can be easily modified to show that $\mu \oplus \lambda$ yields a quasi-isomorphism between $\mathrm{C}_\bullet(\mathcal{A}_{\mathrm{Dunkl}}((\hbar_1))[\![\hbar_2]\!])$ and $\underline{\mathbb{C}}((\hbar_1))[\![\hbar_2]\!][2n] \oplus i_*\underline{\mathbb{C}}_{M^\gamma}((\hbar_1))[\![\hbar_2]\!][2n-2]$ in $\mathrm{D}(\mathrm{PrSh}_{\mathbb{C}((\hbar_1))[\![\hbar_2]\!]}(M))$. Together with Proposition 11, this implies that the trace map $\chi_0 : \mathrm{HH}_\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))[\![\hbar_2]\!]) \to \mathrm{H}^{2n-\bullet}(M, \mathbb{C}((\hbar_1))[\![\hbar_2]\!])^{\mathbb{Z}_2}$ is a surjection. By Proposition 16, one immediately has the first part of the following proposition (the second part being obvious). The reader is reminded that θ and θ_0 have the same meaning as in the proof of Theorem 5.

Proposition 17. There is a $2n$-cycle θ of $\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))$ such that $[\theta_0] = (\hbar_2 \mapsto 0)[\theta] \neq 0$. Further, the following diagram commutes:

$$\begin{tikzcd} \mathrm{HH}_\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))[\![\hbar_2]\!]) \arrow[r] \arrow[d, "\chi_0"] & \mathrm{HH}_\bullet(\mathfrak{A}_{M/\mathbb{Z}_2}((\hbar_1))((\hbar_2))) \arrow[d, "\chi_0"'] \\ \mathrm{H}^{2n-\bullet}(M, \mathbb{C}((\hbar_1))[\![\hbar_2]\!])^{\mathbb{Z}_2} \arrow[r] & \mathrm{H}^{2n-\bullet}(M, \mathbb{C}((\hbar_1))((\hbar_2)))^{\mathbb{Z}_2}. \end{tikzcd}$$ $\square$

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